Dense fluid transport for inelastic hard spheres

V. Garzó* and J. W. Dufty
Department of Physics, University of Florida, Gainesville, Florida 32611
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The revised Enskog theory for inelastic hard spheres is considered as a model for rapid flow granular media at finite densities. A normal solution is obtained via the Chapman-Enskog method for states near the local homogeneous cooling state. The analysis is performed to first order in the spatial gradients, allowing identification of the Navier-Stokes order transport coefficients associated with the heat and momentum fluxes. In addition, the cooling rate is calculated to first order in the gradients and expressed in terms of the transport coefficients. The transport coefficients are determined from linear integral equations analogous to those for elastic collisions. The solubility conditions for these equations are confirmed and the transport coefficients are calculated as explicit functions of the density and restitution coefficient using a Sonine polynomial expansion. The results are not limited to small dissipation. Finally, the analysis is repeated using a simpler kinetic model. Excellent agreement is obtained with the results from the revised Enskog equation. [S1063-651X(99)14405-2]

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I. INTRODUCTION

The analysis of granular media under rapid flow frequently exploits a hydrodynamic description. The essential difference from that for normal fluids is the absence of energy conservation, leading to both obvious and subtle modifications of the usual Navier-Stokes equations for states with small spatial gradients of the hydrodynamic fields. The justification for a hydrodynamic description and a detailed derivation of the form of the transport coefficients remains a topic of interest and controversy. For a low density system these problems have been addressed using the Boltzmann kinetic equation modified to account for inelastic binary collisions. Recently, the Chapman-Enskog solution to the Boltzmann equation, familiar from transport theory for normal gases, has been extended to the inelastic case to obtain the Navier-Stokes equations and detailed expressions for the transport coefficients as functions of the restitution coefficient [1]. The case considered was an idealized gas of smooth, spherical hard spheres with inelastic binary collisions. Preliminary comparisons with direct Monte Carlo simulation of the Boltzmann equation indicate the results are quite accurate, even for strong dissipation [2]. The objective here is to extend this analysis to the revised Enskog kinetic theory (RET) for a description of hydrodynamics and transport at higher densities.

The RET for elastic collisions [3] is known to be an accurate kinetic theory over the entire fluid domain. In fact it describes the crystal phase as well, which may suggest its relevance for granular media undergoing cluster formation. Its generalization to inelastic collisions is straightforward (see, for example, Ref. [4]) and the Chapman-Enskog method can be applied to obtain the Navier-Stokes hydrodynamic equations and the associated transport coefficients. Similar attempts to apply the Boltzmann and Enskog kinetic equations to derive transport coefficients began fifteen years ago [5–8], but the technical difficulties of the analysis entailed approximations that limited their accuracy [1]. The analysis here provides formally exact results for the distribution function and transport coefficients in terms of the solutions to linear integral equations. In this respect it parallels closely the analysis for elastic collisions. As in the latter case, a first Sonine polynomial approximation is used to obtain practical results from this formulation. The approximation technique is known to be accurate for elastic collisions and is expected to have comparable accuracy here as well. The results apply to all degrees of dissipation and over the entire fluid density range. Consequently, it provides the basis for an unambiguous basis for the application of hydrodynamics to granular flow under realistic conditions.

There are several motivations for this study. First, a quantitative test of the hydrodynamic description within the context of kinetic theory is possible by comparing predicted transport properties with those obtained directly from the RET via Monte Carlo simulation. Such simulation methods have been developed and applied in recent years [9] and the results obtained here provide the basis for a test of the standard method for derivation of hydrodynamics. Second, by extending the Boltzmann analysis to high densities comparisons with molecular dynamics simulations become practical. For example, a comparison of the calculated viscosity dependence on both the density and restitution coefficient with that from molecular dynamics would determine the validity (and any limitations) of the kinetic theory and the hydrodynamic descriptions for granular flow. Such a test is essential to address the frequently made speculation that a hydrodynamic and/or kinetic theory description of granular flow is limited to weak dissipation. None of the analysis here supports this latter view, beyond complications of the well-known hydrodynamic instability of the reference cooling state. Finally, accurate predictions from the RET allow tests of simpler kinetic models that have been proposed [10,11] for application to more complex boundary driven states. In summary, the results presented here provide the basis for both practical quantitative applications and fundamental benchmark tests.

The RET and associated macroscopic conservation laws

*Permanent address: Departamento de Física, Universidad de Extremadura, 06071 Badajoz, Spain.

are reviewed in the next section. An overview of the Chapman-Enskog method for solving this equation is given in Sec. III. The results for the distribution function, heat flux, and momentum flux to first order in the spatial gradients are provided, with the details of the calculation appearing in a series of Appendixes. As in the case of elastic collisions these properties are expressed in terms of solutions to a set of linear inhomogeneous integral equations. It is shown explicitly that the necessary solubility conditions are satisfied. In Sec. IV these equations are solved using a lowest order expansion in Sonine polynomials. Explicit expressions for the distribution function and transport coefficients are given and compared to the elastic limit as a function of density and restitution coefficient. In Sec. V the corresponding results from a simple kinetic model are obtained and compared with those from the RET. Excellent agreement is found for a wide range of densities and restitution coefficients. This analysis is summarized and discussed in the last section.

II. RET AND CONSERVATION LAWS

In this section a brief review of the RET kinetic equation and associated conservation laws are presented. The one-particle distribution function $f(\mathbf{r},\mathbf{v},t)$ is determined from [4]

$$\left(\frac{\partial}{\partial t}+\mathbf{v}_1\cdot\boldsymbol{\nabla}_1\right)f(\mathbf{r}_1,\mathbf{v}_1,t,)=J_E[\mathbf{r}_1,\mathbf{v}_1|f(t)], \tag{1}$$

where J_E is the Enskog collision operator

$$J_E[\mathbf{r}_1,\mathbf{v}_1|f(t)]\equiv\sigma^2\int d\mathbf{v}_2\int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})\times\{\alpha^{-2}f^{(2)}(\mathbf{r}_1,\mathbf{r}_1-\boldsymbol{\sigma},\mathbf{v}_1',\mathbf{v}_2',t)-f^{(2)}(\mathbf{r}_1,\mathbf{r}_1+\boldsymbol{\sigma},\mathbf{v}_1,\mathbf{v}_2,t)\}, \tag{2}$$

$$f^{(2)}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{v}_1,\mathbf{v}_2,t)\equiv\chi(\mathbf{r}_1,\mathbf{r}_2|n(t))f(\mathbf{r}_1,\mathbf{v}_1,t,)f(\mathbf{r}_2,\mathbf{v}_2,t). \tag{3}$$

Here, σ is the hard sphere diameter, $\boldsymbol{\sigma}=\sigma\hat{\boldsymbol{\sigma}}$, $\hat{\boldsymbol{\sigma}}$ being a unit vector, Θ is the Heaviside step function, and $\mathbf{g}=\mathbf{v}_1-\mathbf{v}_2$. The primes on the velocities denote the initial values $\{\mathbf{v}_1',\mathbf{v}_2'\}$ that lead to $\{\mathbf{v}_1,\mathbf{v}_2\}$ following a binary collision, $\mathbf{v}_1'=\mathbf{v}_1-\frac{1}{2}(1+\alpha^{-1})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})\hat{\boldsymbol{\sigma}}$, $\mathbf{v}_2'=\mathbf{v}_2+\frac{1}{2}(1+\alpha^{-1})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})\hat{\boldsymbol{\sigma}}$. The parameter α is the coefficient of restitution with $0<\alpha\leqslant 1$. Finally, $\chi[\mathbf{r},\mathbf{r}+\boldsymbol{\sigma}|n(t)]$ is the equilibrium pair correlation function at contact as a functional of the nonequilibrium density field $n(\mathbf{r},t)$ defined by $n(\mathbf{r},t)=\int d\mathbf{v}f(\mathbf{r},\mathbf{v},t)$.

The macroscopic balance equations for density n, momentum $m\mathbf{U}$, and energy $\frac{3}{2}nk_BT$ follow directly from Eq. (1) by multiplying with 1, $m\mathbf{v}_1$, and $mv_1^2/2$ and integrating over $\mathbf{v}_1$,

$$D_t n+n\boldsymbol{\nabla}\cdot\mathbf{U}=0, \tag{4}$$

$$D_t U_i+(mn)^{-1}\partial_j P_{ij}=0, \tag{5}$$

$$D_t T+\frac{2}{3nk_B}(P_{ij}\partial_j U_i+\boldsymbol{\nabla}\cdot\mathbf{q})=-T\zeta, \tag{6}$$

where $D_t=\partial_t+\mathbf{U}\cdot\boldsymbol{\nabla}$ is the material derivative, m is the mass of a particle, k_B is the Boltzmann's constant, $T(\mathbf{r},t)$ is the temperature, and $\mathbf{U}(\mathbf{r},t)$ is the flow velocity. The cooling rate ζ is proportional to $(1-\alpha^2)$ and is due to dissipative collisions. The pressure tensor $\mathsf{P}(\mathbf{r},t)$ and the heat flux $\mathbf{q}(\mathbf{r},t)$ have both ''kinetic'' and ''collisional transfer'' contributions, i.e., $\mathsf{P}=\mathsf{P}^k+\mathsf{P}^c$ and $\mathbf{q}=\mathbf{q}^k+\mathbf{q}^c$. The kinetic contributions are given by

$$\mathsf{P}^k=\int d\mathbf{v}\,m\mathbf{V}\mathbf{V}f(\mathbf{r},\mathbf{v},t),\quad \mathbf{q}^k=\int d\mathbf{v}\frac{1}{2}mV^2\mathbf{V}f(\mathbf{r},\mathbf{v},t), \tag{7}$$

and the collisional transfer contributions are [4]

$$\mathsf{P}^c=\frac{1+\alpha}{4}m\sigma^3\int d\mathbf{v}_1\int d\mathbf{v}_2\int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})\times(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})^2\hat{\boldsymbol{\sigma}}\hat{\boldsymbol{\sigma}}\int_0^1 d\lambda f^{(2)}[\mathbf{r}-(1-\lambda)\boldsymbol{\sigma},\mathbf{r}+\lambda\boldsymbol{\sigma},\mathbf{v}_1,\mathbf{v}_2,t], \tag{8}$$

$$\mathbf{q}^c=\frac{1+\alpha}{4}m\sigma^3\int d\mathbf{v}_1\int d\mathbf{v}_2\int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})^2\times(\mathbf{G}\cdot\hat{\boldsymbol{\sigma}})\hat{\boldsymbol{\sigma}}\int_0^1 d\lambda f^{(2)}[\mathbf{r}-(1-\lambda)\boldsymbol{\sigma},\mathbf{r}+\lambda\boldsymbol{\sigma},\mathbf{v}_1,\mathbf{v}_2,t]. \tag{9}$$

Here, $\mathbf{V}=\mathbf{v}-\mathbf{U}(\mathbf{r},t)$ is the peculiar velocity and $\mathbf{G}=\frac{1}{2}(\mathbf{V}_1+\mathbf{V}_2)$ is the velocity of center of mass. Finally, the cooling rate in Eq. (6) is given by

$$\zeta=(1-\alpha^2)\frac{\beta m\sigma^2}{12n}\int d\mathbf{v}_1\int d\mathbf{v}_2\int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})\times(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})^3 f^{(2)}(\mathbf{r},\mathbf{r}+\boldsymbol{\sigma},\mathbf{v}_1,\mathbf{v}_2,t), \tag{10}$$

with $\beta\equiv 1/(k_BT)$.

The macroscopic balance equations (4)–(6) are exact consequences of the RET and provide the basis for developing a hydrodynamic description. The latter follows from a representation of $\mathsf{P}(\mathbf{r},t)$, $\mathbf{q}(\mathbf{r},t)$, and $\zeta(\mathbf{r},t)$ as explicit functionals of the hydrodynamic fields and their gradients. The resulting closed set of equations constitutes the hydrodynamics for the RET. Since $\mathsf{P}(\mathbf{r},t)$, $\mathbf{q}(\mathbf{r},t)$, and $\zeta(\mathbf{r},t)$ are given above as explicit functionals of $f(\mathbf{r},\mathbf{v},t)$ such a representation can be obtained if a corresponding solution to the RET can be obtained as a function of the fields and their gradients. This is done in the next section using the Chapman-Enskog method.

III. CHAPMAN-ENSKOG EXPANSION

The Chapman-Enskog method [12] assumes the existence of a ''normal'' solution such that all space and time dependence of the distribution function occurs through the hydrodynamic fields

$$f(\mathbf{r},\mathbf{v},t)=f[\mathbf{v}|n(t),T(t),\mathbf{U}(t)]. \tag{11}$$

The notation on the right side indicates a functional dependence on the density, temperature, and flow velocity. When this form is substituted into the RET all space and time de-

rivatives occur through the corresponding derivatives of the hydrodynamic fields. The time derivatives can be expressed in terms of space derivatives using the macroscopic balance equations (4)–(6). In this way the RET becomes an equation to determine the functional dependence of $f[\mathbf{v}|n(t),T(t),\mathbf{U}(t)]$ on the fields which are then determined self-consistently from the macroscopic balance equations. This procedure becomes more practical for states with small spatial gradients. Then the functional dependence can be made local by an expansion of the fields in terms of their gradients, with the distribution function determined perturbatively. To implement this expansion it is convenient to introduce a formal expansion parameter ϵ referred to as the uniformity parameter. It measures the dependence on spatial gradients (e.g., a term of order ϵ is of first order in a hydrodynamic gradient, ϵ^2 is either a product of two first order hydrodynamic gradients or one second degree hydrodynamic gradient). Accordingly the distribution function, collision operator, and time derivative are given by the representations

$$f=f^{(0)}+\epsilon f^{(1)}+\cdots, \quad J_E=J^{(0)}+\epsilon J^{(1)}+\cdots,$$
$$\partial_t=\partial_t^{(0)}+\epsilon\partial_t^{(1)}+\cdots. \tag{12}$$

The coefficients in the time derivative expansion are identified by a representation of the fluxes and the cooling rate in the macroscopic balance equations as a similar series through their definitions as functionals of f.

To zeroth order in ϵ the RET becomes

$$\partial_t^{(0)}f^{(0)}=J^{(0)}[f^{(0)},f^{(0)}], \tag{13}$$

$$J^{(0)}[f^{(0)},f^{(0)}]\equiv\chi\sigma^2\int d\mathbf{v}_2\int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})\times\{\alpha^{-2}f^{(0)}(\mathbf{v}_1')f^{(0)}(\mathbf{v}_2')-f^{(0)}(\mathbf{v}_1)f^{(0)}(\mathbf{v}_2)\}. \tag{14}$$

Here $\chi=\chi[\mathbf{r},\mathbf{r}+\boldsymbol{\sigma}|n(t)]|_{n=n(\mathbf{r},t)}$ is the pair functional evaluated with all density fields at the local point $\mathbf{r}$. The collision operator (14) can be recognized as the Boltzmann operator for inelastic collisions multiplied by this factor χ. The macroscopic balance equations to this order read

$$\partial_t^{(0)}n=0, \quad \partial_t^{(0)}\mathbf{U}=0, \quad T^{-1}\partial_t^{(0)}T=-\zeta^{(0)}, \tag{15}$$

where the cooling rate $\zeta^{(0)}$ is determined by Eq. (10) to zeroth order

$$\zeta^{(0)}=(1-\alpha^2)\frac{\chi\beta m\pi\sigma^2}{24n}\int d\mathbf{v}_1\int d\mathbf{v}_2\times f^{(0)}(\mathbf{v}_1)f^{(0)}(\mathbf{v}_2)|\mathbf{v}_1-\mathbf{v}_2|^3. \tag{16}$$

Since $f^{(0)}$ has the form (11) the time derivative can be evaluated as

$$\partial_t^{(0)}f^{(0)}=-\zeta^{(0)}T\partial_T f^{(0)}=\frac{1}{2}\zeta^{(0)}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)}), \tag{17}$$

where $\boldsymbol{\nabla}_v=\partial/\partial\mathbf{v}$. The second equality follows from dimensional analysis which requires that the temperature dependence of $f^{(0)}$ must occur in the form

$$f^{(0)}=nv_0^{-3}\phi(V/v_0), \quad v_0=\sqrt{2k_BT(\mathbf{r},t)/m}. \tag{18}$$

The dependence on the magnitude of $\mathbf{V}$ follows from the requirement that to zeroth order in the gradients the distribution function must be isotropic with respect to the peculiar velocity. The RET to this order is therefore the solution to

$$\frac{1}{2}\zeta^{(0)}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})=J^{(0)}[f^{(0)},f^{(0)}]. \tag{19}$$

To date an exact solution has not been found. However, a very good approximation can be obtained from an expansion in Sonine polynomials [13]. In all of the following it is presumed that $f^{(0)}$ is known. Since the distribution function is isotropic the zeroth order pressure tensor and heat flux are found from Eqs. (7)–(10) to be

$$P_{ij}^{(0)}=p\,\delta_{ij}, \quad \mathbf{q}^{(0)}=\mathbf{0}, \tag{20}$$

where p is the hydrostatic pressure

$$p=nk_BT+\frac{1+\alpha}{12}m\sigma^3\chi\int d\mathbf{v}_1\int d\mathbf{v}_2\,f^{(0)}(V_1)f^{(0)}(V_2)\times\int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})^2$$
$$=nk_BT\left(1+\frac{1+\alpha}{3}\pi n^*\chi\right). \tag{21}$$

The dimensionless density $n^*=n\sigma^3$ has been introduced.

The analysis is similar to first order in ϵ, although technically more complex. Only the results are presented in this section with the details given in the appendixes. The RET equation to first order is found to be

$$(\partial_t^{(0)}+\mathcal{L})f^{(1)}+\frac{1}{2}\zeta^{(1,1)}(f^{(1)})\mathcal{Q}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})=\mathbf{A}(\mathbf{V})\cdot\boldsymbol{\nabla}\ln T+\mathbf{B}(\mathbf{V})\cdot\boldsymbol{\nabla}\ln n+C_{ij}(\mathbf{V})\times\frac{1}{2}\left(\partial_jU_i+\partial_iU_j-\frac{2}{3}\delta_{ij}\boldsymbol{\nabla}\cdot\mathbf{U}\right)+D(\mathbf{V})\boldsymbol{\nabla}\cdot\mathbf{U}, \tag{22}$$

where $\mathcal{L}$ is the linearized collision operator

$$\mathcal{L}(X)\equiv-\mathcal{Q}(J^{(0)}[f^{(0)},X]+J^{(0)}[X,f^{(0)}]). \tag{23}$$

Here $\mathcal{Q}=1-\mathcal{P}$, where $\mathcal{P}$ is the projection operator onto the space spanned by 1, $\mathbf{V}$, and V^2 [see Eq. (A3) of Appendix A]. The velocity dependence on the right side of Eq. (22) is

$$\mathbf{A}(\mathbf{V}|n,\mathbf{U},T)=\left(\frac{5}{2}f^{(0)}+\frac{1}{2}\mathbf{V}\cdot\boldsymbol{\nabla}_vf^{(0)}\right)\mathbf{V}-\frac{p}{p^k}[\mathbf{V}f^{(0)}+(\beta m)^{-1}\boldsymbol{\nabla}_vf^{(0)}]+\frac{1}{2}\mathcal{K}[\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})], \tag{24}$$

$$\mathbf{B}(\mathbf{V}|n,\mathbf{U},T)=-\frac{p}{p^k}\left(1+n\partial_n\ln\frac{p}{p^k}\right)[\mathbf{V}f^{(0)}+(\beta m)^{-1}\boldsymbol{\nabla}_v f^{(0)}]-\left(1+\frac{1}{2}n\partial_n\ln\chi\right)\boldsymbol{\mathcal{K}}[f^{(0)}], \tag{25}$$

$$C_{ij}(\mathbf{V}|n,\mathbf{U},T)=V_i(\boldsymbol{\nabla}_{v_j}f^{(0)})+\mathcal{K}_i[\boldsymbol{\nabla}_{v_j}f^{(0)}], \tag{26}$$

$$D(\mathbf{V})=\frac{1}{3}\mathcal{K}_i[\boldsymbol{\nabla}_{v_i}f^{(0)}]+\frac{p-p^k}{3p^k}(2-3\alpha)\mathcal{Q}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)}). \tag{27}$$

Here, $\boldsymbol{\nabla}_{v_i}=\partial/\partial v_i$, $p^k=nk_BT$ and the linear operator $\boldsymbol{\mathcal{K}}[X]$ is defined by

$$\boldsymbol{\mathcal{K}}[X]=\mathcal{Q}\chi\sigma^2\int d\mathbf{v}_2\int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})\times(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})\boldsymbol{\sigma}\{\alpha^{-2}f^{(0)}(\mathbf{v}_1')X(\mathbf{v}_2')+f^{(0)}(\mathbf{v}_1)X(\mathbf{v}_2)\}. \tag{28}$$

The functional $\zeta^{(1,1)}(f^{(1)})$ is related to $\zeta^{(1)}$ by [see, Eq. (A23)]

$$\zeta^{(1)}=-(1-\alpha)\frac{p-p^k}{p^k}\boldsymbol{\nabla}\cdot\mathbf{U}+\zeta^{(1,1)}(f^{(1)}), \tag{29}$$

$$\zeta^{(1,1)}(f^{(1)})\equiv(1-\alpha^2)\frac{\beta m\pi\sigma^2\chi}{12n}\int d\mathbf{V}_1\int d\mathbf{V}_2\times f^{(0)}(V_1)f^{(1)}(V_2)g^3. \tag{30}$$

The solution to Eq. (22) is of the form

$$f^{(1)}=\boldsymbol{\mathcal{A}}\cdot\nabla\ln T+\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\nabla}\ln n+\mathcal{C}_{ij}\frac{1}{2}\left(\partial_jU_i+\partial_iU_j-\frac{2}{3}\delta_{ij}\boldsymbol{\nabla}\cdot\mathbf{U}\right)+\mathcal{D}\boldsymbol{\nabla}\cdot\mathbf{U}. \tag{31}$$

Substitution of this into Eq. (22) gives the integral equations to determine $\boldsymbol{\mathcal{A}}(\mathbf{V})$, $\boldsymbol{\mathcal{B}}(\mathbf{V})$, $\mathcal{C}_{ij}(\mathbf{V})$, and $\mathcal{D}(\mathbf{V})$

$$\left(-\zeta^{(0)}T\partial_T+\mathcal{L}-\frac{\zeta^{(0)}}{2}\right)\boldsymbol{\mathcal{A}}=\mathbf{A}, \tag{32}$$

$$(-\zeta^{(0)}T\partial_T+\mathcal{L})\boldsymbol{\mathcal{B}}=[\mathbf{B}+(1+n\partial_n\ln\chi)\zeta^{(0)}\boldsymbol{\mathcal{A}}], \tag{33}$$

$$(-\zeta^{(0)}T\partial_T+\mathcal{L})\mathcal{C}_{ij}=C_{ij}, \tag{34}$$

$$(-\zeta^{(0)}T\partial_T+\mathcal{L})\mathcal{D}+\frac{1}{2}\zeta^{(1,1)}(\mathcal{D})\mathcal{Q}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})=D. \tag{35}$$

Use has been made of the fact that $\zeta^{(1,1)}(f^{(1)})$ is a scalar and therefore couples only to $\mathcal{D}$. The lowest order distribution $f^{(0)}$ is a functional of the exact nonequilibrium fields. Consequently, its moments with respect to 1, $\mathbf{V}$, and V^2 are the same as those for the exact distribution, i.e., $\mathcal{P}f^{(0)}=\mathcal{P}f$. This implies that $f^{(1)}$ lies entirely in the orthogonal subspace so acceptable solutions to Eqs. (32)–(34) must obey

$$\mathcal{P}\begin{pmatrix}\boldsymbol{\mathcal{A}}\\ \boldsymbol{\mathcal{B}}\\ \mathcal{C}_{ij}\\ \mathcal{D}\end{pmatrix}=0. \tag{36}$$

There are additional necessary conditions for the solution to these equations to exist, the Fredholm alternative [14]. These ''solubility conditions'' are proved in Appendix B.

With the distribution function $f^{(1)}$ determined, the heat and momentum fluxes can be calculated to first order in the spatial gradients from Eqs. (7)–(9). The details of the calculations are given in Appendixes C and D with the results

$$P_{ij}^{(1)}=-\eta\left(\partial_jU_i+\partial_iU_j-\frac{2}{3}\delta_{ij}\boldsymbol{\nabla}\cdot\mathbf{U}\right)-\gamma\delta_{ij}\boldsymbol{\nabla}\cdot\mathbf{U},$$

$$\mathbf{q}^{(1)}=-\kappa\boldsymbol{\nabla}T-\mu\boldsymbol{\nabla}n, \tag{37}$$

where η is the shear viscosity, γ is the bulk viscosity, κ is the thermal conductivity, and μ is an additional transport coefficient not present for elastic collisions

$$\eta=\eta^k\left[1+\frac{2\pi n^*\chi(1+\alpha)}{15}\right]+\frac{3}{5}\gamma, \tag{38}$$

$$\gamma=\frac{\pi mn^*n\chi\sigma v_0(1+\alpha)}{18}I_\gamma(\alpha), \tag{39}$$

$$\kappa=\kappa^k\left[1+\frac{\pi n^*\chi(1+\alpha)}{5}\right]+\frac{\pi mn^*\chi n\sigma v_0^3(1+\alpha)}{48T}I_\kappa(\alpha), \tag{40}$$

$$\mu=\mu^k\left[1+\frac{\pi n^*\chi(1+\alpha)}{5}\right]. \tag{41}$$

Here the superscript k denotes the contributions from the kinetic parts of the fluxes. The kinetic parts of the transport coefficients in these results are expressed in terms of the solution to the integral equations as

$$\eta^k=-\frac{1}{10}\int d\mathbf{V}\,D_{ij}(\mathbf{V})\mathcal{C}_{ij}(\mathbf{V}),\quad \gamma^k=0, \tag{42}$$

$$\kappa^k=-\frac{1}{3T}\int d\mathbf{V}\,\mathbf{S}(\mathbf{V})\cdot\boldsymbol{\mathcal{A}}(\mathbf{V}),$$

$$\mu^k=-\frac{1}{3n}\int d\mathbf{V}\,\mathbf{S}(\mathbf{V})\cdot\boldsymbol{\mathcal{B}}(\mathbf{V}), \tag{43}$$

where $D_{ij}(\mathbf{V})$ and $\mathbf{S}(\mathbf{V})$ are

$$D_{ij}(\mathbf{V})=m\left(V_iV_j-\frac{1}{3}V^2\delta_{ij}\right),\quad \mathbf{S}(\mathbf{V})=\mathbf{V}\left(\frac{1}{2}mV^2-\frac{5}{2}k_BT\right). \tag{44}$$

The dimensionless integrals $I_\gamma(\alpha)$ and $I_\kappa(\alpha)$ in Eqs. (39) and (40) are

$$I_\gamma(\alpha)=\frac{1}{n^2 v_0}\int d\mathbf{V}_1\int d\mathbf{V}_2\; f^{(0)}(V_1)f^{(0)}(V_2)g, \quad (45)$$

$$I_\kappa(\alpha)=\frac{1}{n^2 v_0^3}\int d\mathbf{V}_1\int d\mathbf{V}_2\; f^{(0)}(V_1)f^{(0)}(V_2) \times\left[g^{-1}(\mathbf{g}\cdot\mathbf{G})^2+gG^2+\frac{3}{2}g(\mathbf{g}\cdot\mathbf{G})+\frac{1}{4}g^3\right]. \quad (46)$$

In addition to the fluxes, the cooling rate ζ can be calculated to first order in the gradients from Eqs. (29) and (30)

$$\zeta^{(1)}=(1-\alpha)\frac{p-p^k}{p^k}\left[-1+\frac{\beta m}{4n^2\sigma}\int d\mathbf{V}_1\int d\mathbf{V}_2 \times f^{(0)}(V_1)\mathcal{D}(V_2)g^3\right]\boldsymbol{\nabla}\cdot\mathbf{U}. \quad (47)$$

This completes the results of the Chapman-Enskog solution to first order in the spatial gradients. The distribution function $f=f^{(0)}+f^{(1)}$ with $f^{(0)}$ given by the solution to Eq. (19) and $f^{(1)}$ given by Eq. (31) is exact to this order in the gradients, as are the fluxes (37). These results require solution to the linear equations (32)–(34). An approximate solution, known to be accurate for elastic collisions, is given in the next section. For that purpose it is convenient to use an equivalent exact expression for the kinetic parts of the transport coefficients (see Appendix C)

$$\eta^k=\frac{n}{\beta}\left(\nu_\eta-\frac{1}{2}\zeta^{(0)}\right)^{-1}\left[1-\frac{1}{15}(1+\alpha)(1-3\alpha)\pi n^*\chi\right], \quad (48)$$

$$\kappa^k=\frac{5nk_B}{2\beta m}(\nu_\kappa-2\zeta^{(0)})^{-1} \times\left\{1+\frac{1}{2}\left(1+\frac{p}{p^k}\right)c^*+\frac{1}{10}\pi n^*\chi(1+\alpha)^2 \times\left[2\alpha-1+\left(\frac{1}{2}(1+\alpha)-\frac{5}{3(1+\alpha)}\right)c^*\right]\right\}, \quad (49)$$

$$\mu^k=\frac{15}{2\beta^2 m}(2\nu_\mu-3\zeta^{(0)})^{-1}\left\{(1+n\partial_n\ln\chi)\zeta^{(0)}(\kappa^k/\kappa_0) +\frac{p}{3p^k}\left(1+n\partial_n\ln\frac{p}{p^k}\right)c^*-\frac{2}{15}\pi n^*\chi(1+\alpha) \times\left(1+\frac{1}{2}n\partial_n\ln\chi\right)\left[\alpha(1-\alpha)+\frac{1}{4}\left(\alpha(1-\alpha)+\frac{4}{3}\right)c^*\right]\right\}, \quad (50)$$

with the definitions

$$\nu_\eta=\frac{\int d\mathbf{V}D_{ij}(\mathbf{V})\mathcal{L}C_{ij}(\mathbf{V})}{\int d\mathbf{V}D_{ij}(\mathbf{V})\mathcal{C}_{ij}(\mathbf{V})},$$

$$\nu_\kappa=\frac{\int d\mathbf{V}\,\mathbf{S}(\mathbf{V})\cdot\mathcal{L}\boldsymbol{\mathcal{A}}(\mathbf{V})}{\int d\mathbf{V}\,\mathbf{S}(\mathbf{V})\cdot\boldsymbol{\mathcal{A}}(\mathbf{V})}, \quad \nu_\mu=\frac{\int d\mathbf{V}\,\mathbf{S}(\mathbf{V})\cdot\mathcal{L}\boldsymbol{\mathcal{B}}(\mathbf{V})}{\int d\mathbf{V}\,\mathbf{S}(\mathbf{V})\cdot\boldsymbol{\mathcal{B}}(\mathbf{V})}. \quad (51)$$

Further, $\kappa_0=15k_B\eta_0/4m$ and $\eta_0=5m^{1/2}/16\sigma^2(\pi\beta)^{1/2}$ are the low density values of the thermal conductivity and the shear viscosity in the elastic limit, respectively. The constant $c^*(\alpha)$ is related to the fourth moment of $f^{(0)}$ by

$$c^*(\alpha)=\frac{8}{15}\left[\left(\frac{m\beta}{2}\right)^2\frac{1}{n}\int d\mathbf{V}f^{(0)}(V)V^4-\frac{15}{4}\right]. \quad (52)$$

It is seen that $c^*(\alpha)$ vanishes if $f^{(0)}$ is replaced by the Maxwellian distribution, and hence it is a measure of the deviation of the reference state from that for a gas with elastic collisions.

IV. SONINE POLYNOMIAL APPROXIMATION

More explicit results require determination of $f^{(0)}$ and the solutions $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$, and $\mathcal{C}_{ij}$ to the linear integral equations (32)–(34). It is useful to represent these quantities as an expansion in a complete set of polynomials and generate approximations by truncating the expansion. In practice the leading term in these expansions provides a very accurate description over the full range of dissipation and density. The determination of $f^{(0)}$ to leading order in the Sonine polynomial expansion has been discussed elsewhere [13] so only the result is quoted here

$$f^{(0)}(V)=f_M(V)\left[1+\frac{1}{4}c^*E(V)\right], \quad (53)$$

$$E(V)=\left(\frac{1}{2}\beta m V^2\right)^2-\frac{5}{2}\beta m V^2+\frac{15}{4}. \quad (54)$$

The constant c^* is the same as that of Eq. (52). It is determined by substituting Eq. (53) into the lowest order RET equation (19) and retaining linear terms in the first Sonine polynomial to get

$$c^*(\alpha)=\frac{32(1-\alpha)(1-2\alpha^2)}{81-17\alpha+30\alpha^2(1-\alpha)}. \quad (55)$$

Comparison of this prediction with direct Monte Carlo simulation of the kinetic equation shows a very good agreement even for relatively large dissipation [15]. The tensorial properties of $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$, $\mathcal{C}_{ij}$, and $\mathcal{D}$ are obvious and traditionally the scalar magnitudes have been expanded in terms of Sonine polynomials. For elastic collisions truncation of the series at the leading first Sonine polynomial approximation, taking

into account the requirement (36), gives very good results (accurate to within a few percent). In the present case this approximation is

$$\begin{pmatrix} \mathcal{A}(\mathbf{V}) \\ \mathcal{B}(\mathbf{V}) \\ \mathcal{C}_{ij}(\mathbf{V}) \\ \mathcal{D}(\mathbf{V}) \end{pmatrix} \to f_M(V) \begin{pmatrix} c_{\mathcal{A}}\mathbf{S}(\mathbf{V}) \\ c_{\mathcal{B}}\mathbf{S}(\mathbf{V}) \\ c_{\mathcal{C}}D_{ij}(\mathbf{V}) \\ c_{\mathcal{D}}E(\mathbf{V}) \end{pmatrix},$$

$$f_M(V) = n\left(\frac{v_0^2}{\pi}\right)^{3/2} e^{-(V/v_0)^2}. \tag{56}$$

The factor $f_M(V)$ occurs since these polynomials are defined relative to a Gaussian scalar product. The coefficients are the projections of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}_{ij}$, and $\mathcal{D}$ along $\mathbf{S}(\mathbf{V})$, $D_{ij}(\mathbf{V})$, and $E(V)$

$$\begin{pmatrix} c_{\mathcal{A}} \\ c_{\mathcal{B}} \end{pmatrix} = \frac{2m\beta^3}{15n}\int d\mathbf{V} \begin{pmatrix} \mathcal{A}(\mathbf{V})\cdot\mathbf{S}(\mathbf{V}) \\ \mathcal{B}(\mathbf{V})\cdot\mathbf{S}(\mathbf{V}) \end{pmatrix} = \begin{pmatrix} -\dfrac{2m\beta^2}{5nk_B}\kappa^k \\ -\dfrac{2mn\beta^3}{5n}\mu^k \end{pmatrix}, \tag{57}$$

$$c_{\mathcal{C}} = \frac{\beta^2}{10n}\int d\mathbf{V}\mathcal{C}_{ij}(\mathbf{V})D_{ij}(\mathbf{V}) = -\frac{\beta^2}{n}\eta^k, \tag{58}$$

$$c_{\mathcal{D}} = \frac{2}{15n}\int d\mathbf{V}\mathcal{D}(\mathbf{V})E(V), \tag{59}$$

where use has been made of the definitions (42) and (43). The distribution function $f^{(1)}$ in this approximation is obtained from Eq. (31)

$$f^{(1)} \to -f_M\left[\frac{2m\beta^3}{5n}(\kappa^k\boldsymbol{\nabla}T + \mu^k\boldsymbol{\nabla}n)\cdot\mathbf{S}(\mathbf{V}) + \frac{\beta^2}{n}\eta^k\frac{1}{2}\left(\partial_i U_j + \partial_j U_i - \frac{2}{3}\delta_{ij}\boldsymbol{\nabla}\cdot\mathbf{U}\right)D_{ij}(\mathbf{V}) - c_{\mathcal{D}}E(V)\boldsymbol{\nabla}\cdot\mathbf{U}\right]. \tag{60}$$

To evaluate the kinetic parts of the transport coefficients the forms (48)–(50) can be used, with the frequencies $\nu_\eta(\alpha)$, $\nu_\kappa(\alpha)$, and $\nu_\mu(\alpha)$ determined from Eq. (51) using the approximation (56)

$$\nu_\eta = \frac{\int d\mathbf{V}D_{ij}\mathcal{L}f_M D_{ij}}{\int d\mathbf{V}\, f_M D_{ij}D_{ij}}, \quad \nu_\kappa = \nu_\mu = \frac{\int d\mathbf{V}\,\mathbf{S}\cdot\mathcal{L}f_M\mathbf{S}}{\int d\mathbf{V}\, f_M\mathbf{S}\cdot\mathbf{S}}. \tag{61}$$

The calculation of all properties now has been reduced to quadratures. An outline of the integrations involved are given in Appendix E while the contributions to the cooling rate are obtained in Appendix F. The results are summarized in Table I. The transport coefficients have been reduced by η_0 and κ_0. The cooling rate ζ and the frequencies ν_η and $\nu_\kappa = \nu_\mu$ have been reduced by the characteristic collision frequency $\nu_0 = p^k/\eta_0$. The resulting dimensionless variables are

$$\eta^* = \eta/\eta_0, \quad \gamma^* = \gamma/\eta_0, \quad \kappa^* = \kappa/\kappa_0, \quad \mu^* = n\mu/T\kappa_0, \tag{62}$$

$$p^* = p/p^k, \quad \zeta^* = \zeta/\nu_0, \quad \nu_\eta^* = \nu_\eta/\nu_0, \quad \nu_\kappa^* = \nu_\mu^* = \nu_\kappa/\nu_0. \tag{63}$$

Also shown in this table is the Carnahan-Starling approximation for χ as a function of n^*. It is easy to check that all results presented in Table I have the correct elastic limit for $\alpha = 1$ [12].

TABLE I. Summary of the main results of the paper.

$$f = f_M\left\{1 + \tfrac{1}{4}c^*E(V) - \frac{2m\beta^3}{5n}(\kappa^k\boldsymbol{\nabla}T + \mu^k\boldsymbol{\nabla}n)\cdot\mathbf{S}(\mathbf{V}) - \frac{\beta^2}{n}\eta^k\tfrac{1}{2}(\partial_i U_j + \partial_j U_i - \tfrac{2}{3}\delta_{ij}\boldsymbol{\nabla}\cdot\mathbf{U})D_{ij}(\mathbf{V}) + c_{\mathcal{D}}E(V)\boldsymbol{\nabla}\cdot\mathbf{U}\right\},$$

$$\eta^* = \eta^{k*}[1 + \tfrac{2}{15}\pi n^*\chi(1+\alpha)] + \tfrac{3}{5}\gamma^*,$$

$$\eta^{k*} = (\nu_\eta^* - \tfrac{1}{2}\zeta^{(0)*})^{-1}[1 - \tfrac{1}{15}(1+\alpha)(1-3\alpha)\pi n^*\chi],$$

$$\gamma^* = \tfrac{32}{45}\pi n^{*2}\chi(1+\alpha)(1-\tfrac{1}{32}c^*),$$

$$\kappa^* = \kappa^{k*}[1 + \tfrac{1}{5}\pi n^*\chi(1+\alpha)] + \tfrac{64}{225}\pi n^{*2}\chi(1+\alpha)(1+\tfrac{7}{32}c^*),$$

$$\kappa^{k*} = \tfrac{2}{3}(\nu_\kappa^* - 2\zeta^{(0)*})^{-1}\left(1 + \tfrac{1}{2}(1+p^*)c^* + \tfrac{1}{10}\pi n^*\chi(1+\alpha)^2 \times\left\{2\alpha - 1 + \left[\tfrac{1}{2}(1+\alpha) - \frac{5}{3(1+\alpha)}\right]c^*\right\}\right),$$

$$\mu^* = \mu^{k*}[1 + \tfrac{1}{5}\pi n^*\chi(1+\alpha)],$$

$$\mu^{k*} = 2(2\nu_\mu^* - 3\zeta^{(0)*})^{-1}\left\{(1 + n\partial_n\ln\chi)\zeta^{(0)*}\kappa^{k*} + \frac{p^*}{3}(1 + n\partial_n\ln p^*)c^* - \tfrac{2}{15}\pi n^*\chi(1 + \tfrac{1}{2}n\partial_n\ln\chi)(1+\alpha)\{\alpha(1-\alpha) + \tfrac{1}{4}[\tfrac{4}{3} + \alpha(1-\alpha)]c^*\}\right\},$$

$$c_{\mathcal{D}} = \frac{1}{\nu_0}\left[\tfrac{1}{2}\zeta^{(0)*} + \nu_\gamma^* + \frac{5c^*}{64}(1 + \tfrac{3}{64}c^*)\chi(1-\alpha^2)\right]^{-1} \times[\tfrac{2}{45}\lambda\pi n^*\chi + (p^*-1)(\tfrac{2}{3} - \alpha)c^*],$$

$$p^* = 1 + \frac{1+\alpha}{3}\pi n^*\chi, \quad \chi = (1 - \tfrac{1}{12}\pi n^*)(1 - \tfrac{1}{6}\pi n^*)^{-3},$$

$$\zeta^* = \zeta^{(0)*} + \zeta^{(1)*}, \quad \zeta^{(0)*} = \tfrac{5}{12}\chi(1-\alpha^2)(1 + \tfrac{3}{32}c^*),$$

$$\zeta^{(1)*} = \left[-\frac{1}{\nu_0}(1-\alpha)(p^*-1) + \tfrac{5}{32}(1-\alpha^2)(1 + \tfrac{3}{64}c^*)\chi c_{\mathcal{D}}\right]\boldsymbol{\nabla}\cdot\mathbf{U},$$

$$c^* = 32(1-\alpha)(1-2\alpha^2)[81 - 17\alpha + 30\alpha^2(1-\alpha)]^{-1},$$

$$\lambda = \tfrac{3}{8}\left[(1-\alpha)(5\alpha^2 + 4\alpha - 1) + \frac{c^*}{12}(159\alpha + 3\alpha^2 - 19\alpha - 15\alpha^3)\right] \times(1+\alpha),$$

$$\nu_\eta^* = \chi[1 - \tfrac{1}{4}(1-\alpha)^2][1 - \tfrac{1}{64}c^*],$$

$$\nu_\kappa^* = \nu_\mu^* = \tfrac{1}{3}(1+\alpha)\chi\left[1 + \tfrac{33}{16}(1-\alpha) + \frac{19 - 3\alpha}{1024}c^*\right],$$

$$\nu_\gamma^* = \frac{1+\alpha}{48}\chi\left[128 - 96\alpha + 15\alpha^2 - 15\alpha^3 + \frac{c^*}{64}(15\alpha^3 - 15\alpha^2 + 498\alpha - 434)\right].$$

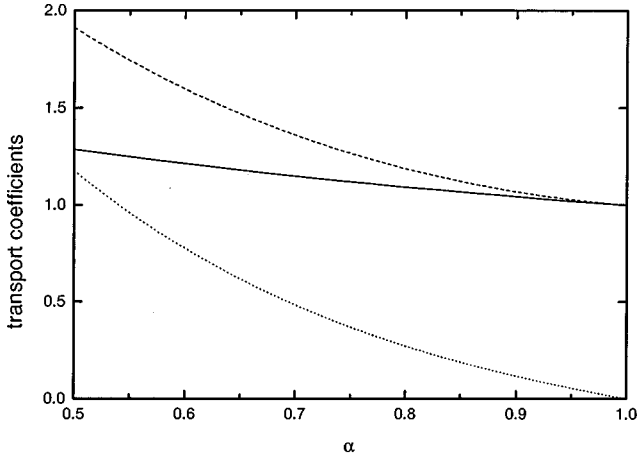

FIG. 1. Plot of the low density transport coefficients as a function of the restitution coefficient α: shear viscosity $\eta^*(\alpha,0)$ (solid line), thermal conductivity $\kappa^*(\alpha,0)$ (dashed line), and $\mu^*(\alpha,0)$ (dotted line).

The transport coefficients for granular flow of dense, weakly inelastic hard spheres were studied by Lun *et al.* [5] by using an approximate moment method to solve the Enskog kinetic equation. However, the method does not include all contributions to leading order in $(1-\alpha)$, and neglects completely those from $\zeta^*(\alpha)$ and $c^*(\alpha)$. It also gives $\mu(\alpha)=0$ in contrast to the finite result found here. In general, Lun's results show important quantitative discrepancies with those derived here. For example, at $\alpha=0.6$ and $n^*=0.2$, the discrepancies are $\sim 11\%$ for the reduced shear viscosity η^* and $\sim 73\%$ for the reduced thermal conductivity κ^*. Figure 1 shows $\eta^*(\alpha,n^*=0)$, $\kappa^*(\alpha,n^*=0)$, and $\mu^*(\alpha,n^*=0)$ as functions of α in the low density limit. These are the same as those given in Ref. [1]. Recent Monte Carlo simulations of the dilute shear viscosity confirm the accuracy of this calculation [2]. Figures 2–4 show $\eta^*(\alpha,n^*)/\eta^*(\alpha,0)$, $\kappa^*(\alpha,n^*)/\kappa^*(\alpha,0)$, and $\mu^*(\alpha,n^*)/\mu^*(\alpha,0)$ as functions of the density for $\alpha=1$, 0.8, and 0.6. Also shown in Figs. 2–4 are the results from the kinetic model discussed in Sec. V [16]. In general, all transport coefficients are increasing functions of the density at sufficiently large α, whose slopes decrease with decreasing α (i.e., greater dissipation).

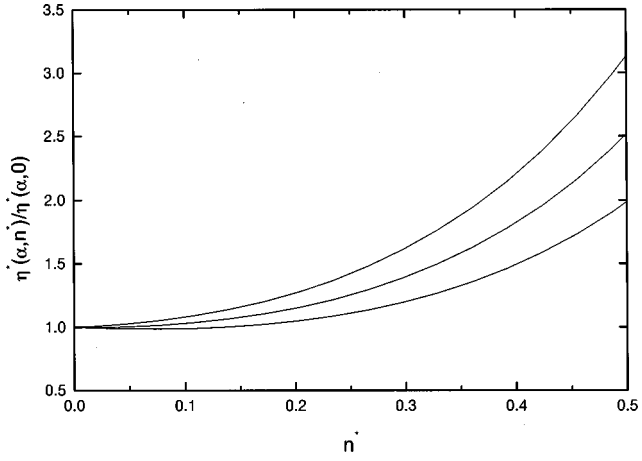

FIG. 2. Plot of the reduced shear viscosity $\eta^*(\alpha,n^*)/\eta^*(\alpha,0)$ as a function of the reduced density n^* as obtained from the Sonine expansion of the RET. From the top to the bottom, we have considered the values of the restitution coefficient $\alpha=1$, 0.8, and 0.6.

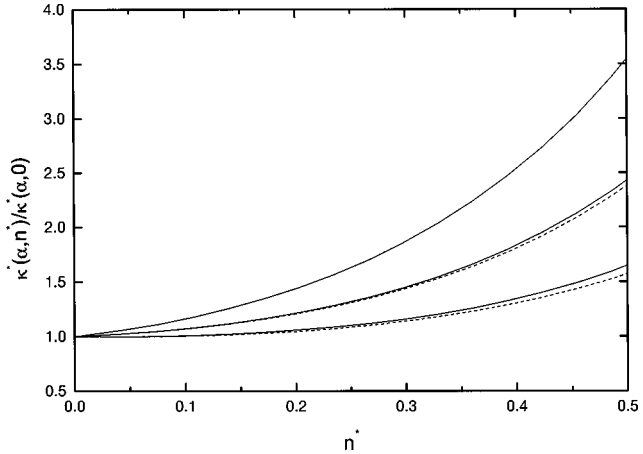

FIG. 3. Plot of the reduced thermal conductivity $\kappa^*(\alpha,n^*)/\kappa^*(\alpha,0)$ as a function of the reduced density n^* as obtained from the Sonine expansion of the RET (solid line) and from the kinetic model equation discussed in Sec. V (dashed line). From the top to the bottom, $\alpha=1$, 0.8, and 0.6.

V. KINETIC MODEL

The Chapman-Enskog solution to the RET described here for states with small spatial gradients is technically difficult but straightforward. For more complex states driven far from equilibrium the RET becomes intractable. In these cases it is useful to have kinetic models with the same qualitative features as the RET but with a mathematically simpler structure. The prototype example for this approach is the Bhatnagar-Gross-Krook (BGK) kinetic model for the Boltzmann equation [17]. Generalizations to the RET for both elastic [18] and inelastic [4,10,11] cases have been discussed recently and applied with excellent success to shear flow far from equilibrium. The objective is to test the most sophisticated of these models [11] for the prediction of the transport proper-

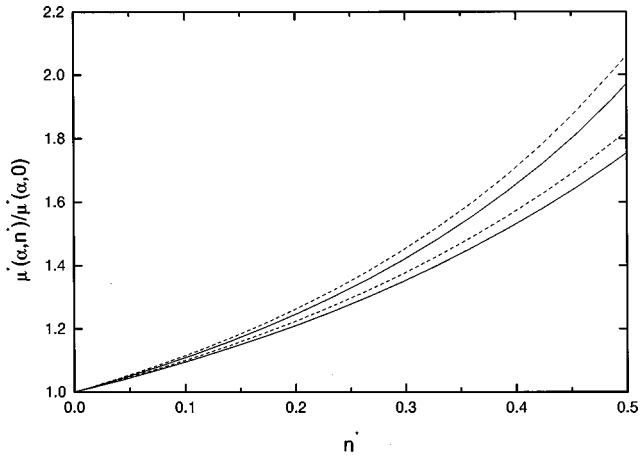

FIG. 4. Plot of the reduced transport coefficient $\mu^*(\alpha,n^*)/\mu^*(\alpha,0)$ as a function of the reduced density n^* as obtained from the Sonine expansion of the RET (solid line) and from the kinetic model equation (dashed line). From the top to the bottom, $\alpha=0.8$ and 0.6.

ties calculated from the RET in the previous sections. The detailed analysis leading to the model kinetic equation can be found in Ref. [11] and will not be repeated here. The model kinetic equation is

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla}\right)f=\mathcal{P}J_E[f]+\mathcal{Q}\Big[-\nu(f-f^{(0)})+(\mathbf{X}\cdot\mathbf{S}+Y_{ij}D_{ij})f^{(0)}+\frac{1}{2}\zeta\boldsymbol{\nabla}_v\cdot(\mathbf{V}f)\Big]. \tag{64}$$

Equation (64) has the same form as Eq. (1) except that the RET collision operator has been replaced by the terms on the right side of Eq. (64). The first term is the exact projection of the collision operator into the subspace spanned by 1, $\mathbf{V}$, and V^2 [see Eq. (A2) of Appendix A]. The second term represent an approximation to the collision operator in the orthogonal subspace, $\mathcal{Q}=1-\mathcal{P}$. It differs from the corresponding kinetic model for elastic collisions [18] only by the last term of the brackets which accounts for the cooling. The functions $\mathbf{S}(\mathbf{V})$ and $D_{ij}(\mathbf{V})$ are given by Eq. (44) while $\mathbf{X}$ and Y_{ij} are defined by

$$\mathbf{X}=\frac{\int d\mathbf{V}\mathbf{S}\,J_E[f^{(0)}]}{\int d\mathbf{V}\,f^{(0)}\mathbf{S}\cdot\mathbf{S}},\quad Y_{ij}=\frac{\int d\mathbf{V}\,D_{ij}J_E[f^{(0)}]}{\int d\mathbf{V}\,f^{(0)}D_{k\ell}D_{k\ell}}. \tag{65}$$

The Chapman-Enskog solution to the kinetic model equation proceeds in the same way as described in Sec. III. To lowest order in the gradients $f^{(0)}$ is again the solution, and $f^{(1)}$ is determined from Eq. (22) with only the replacement

$$\mathcal{L}f^{(1)}\to\nu f^{(1)}-\frac{\zeta^{(0)}}{2}\nabla_v\cdot(\mathbf{V}f^{(1)}). \tag{66}$$

All results of Table I are the same except now

$$\nu_\eta^*=\nu^*+\zeta^{(0)*} \tag{67}$$

and

$$\nu_\kappa^*=\nu_\mu^*=\nu^*+\frac{3}{2}\zeta^{(0)*}, \tag{68}$$

with $\nu^*\equiv\nu/\nu_0$. Since the kinetic model has only one free parameter ν^* only one of these frequencies can be reproduced exactly by the model. For example, the choice ν^* equal to $\nu_\eta^*-\zeta^{(0)*}$ will yield the exact shear and bulk viscosities, but the thermal conductivity and μ will be approximate (the Prandtl number at low density is 1 instead of 2/3). This limitation of the one parameter kinetic model is well known and occurs for the case of elastic collisions as well. It is remarkable that the kinetic model reproduces exactly all other features of transport, both the dependence on dissipation and on the density. This dependence is shown in Figs. 2–4 for different values of the restitution coefficient. Excellent agreement with the Enskog results is found over a wide range of values for n^* and α.

VI. DISCUSSION

The RET kinetic theory for hard spheres with elastic collisions provides the basis for an accurate description of transport over a wide range of densities, as confirmed by both Monte Carlo and molecular dynamics simulations. The objective of the present work has been to extend the application of this equation, with appropriate modifications, to the case of inelastic collisions. The transport processes considered are those for a fluid with small spatial gradients of the hydrodynamic fields. The macroscopic balance equations for mass, momentum, and energy [or, equivalently, for number density, flow velocity, and temperature, as given in Eqs. (4)–(6)] follow directly from moments of the RET with respect to 1, $\mathbf{v}$, and v^2. These equations include the unknown pressure tensor, heat flux, and cooling rate which are given as functionals of the distribution function f. The closed set of hydrodynamic equations are obtained when a solution to the RET is known as a functional of the hydrodynamic fields. Here, this solution is obtained perturbatively for small spatial gradients using the Chapman-Enskog method well known from the case of elastic collisions. The distribution function has been calculated through first order in the gradients. Use of this in the functionals for the pressure tensor, heat flux, and cooling rate provides a representation of these as linear combinations of the gradients. The coefficients in these expressions are the transport coefficients which are functions of the density and restitution coefficient. Use of these first order results in the macroscopic balance equations provides the hydrodynamic equations which are the basis for a fluid dynamics description of granular flow. This work extends a recent corresponding analysis for a low density granular gas [1]. The following comments are offered for context and clarification.

The distribution function f is given in terms of a reference function $f^{(0)}$ which is the solution to Eq. (19). While the exact solution to this equation has not been obtained to date, it is easy to verify that it differs from the Maxwellian for elastic collisions. This reference function depends on space and time only through the hydrodynamic fields. In particular, for both elastic and inelastic cases, the reference state is time dependent. However, for the inelastic case the time scale for the temperature is set by the cooling rate $\zeta^{(0)}$ rather than the spatial gradients. This means that within the hydrodynamic description there can be two well-separated time scales, for weak gradients and strong dissipation. This should not be confused with the separation of *both* hydrodynamic time scales from the shorter kinetic excitations in the solution to RET. Investigations to date [11] suggest that the kinetic excitations remain isolated from the hydrodynamic excitations so that the later dominate on sufficiently large time scales, just as in the elastic case.

The deviations of f from $f^{(0)}$ are proportional to spatial gradients of the temperature, density, and flow velocity. The coefficients of each gradient are functions of the velocity which are determined from linear inhomogeneous integral equations. In the elastic case these coefficients are nonzero only for the temperature gradient and the traceless part of the velocity gradient. For inelastic collisions there are two additional terms, one proportional to the density gradient and one proportional to the divergence of the velocity field. The

former gives rise to an additional term in the heat flux, while the latter provides an additional term in the cooling rate. The latter provides a correction to the results of Ref. [7]. The additional term of the heat flux due to the density gradient has been recognized in earlier work.

The form of $f^{(0)}$ and the solutions to the linear integral equations to determine f and the transport coefficients were determined approximately using a first Sonine polynomial approximation. This is known to be accurate in the elastic limit and to give an excellent approximation for $f^{(0)}$. Preliminary comparisons of the resulting viscosity with Monte Carlo simulations at low density also confirm the accuracy of this method [2]. The method does not make any *a priori* limitation on the degree of dissipation so it is expected that the dependence of the shear viscosity, bulk viscosity, thermal conductivity, μ, and cooling rate on both n^* and α is well described.

A primary outcome of these calculations is a form for the hydrodynamic equations with all parameters given explicitly. The heat and momentum fluxes are determined to first order in the gradients. Since these occur as divergences in the hydrodynamic equations their contributions are of second order in the gradients. In contrast, the cooling rate is determined only to first order, showing a contribution proportional to $\nabla\cdot\mathbf{U}$ and given by $\zeta^{(1)*}$ in Table I. The second term in the brackets of that expression does not appear to have predicted before. In principle, the Navier-Stokes order hydrodynamics for inelastic collisions requires going one order further in the Chapman-Enskog expansion (Burnett order) to obtain the cooling rate to second order. This has been done in the low density limit where it is found that the corrections are very small [1]. Consequently, it is likely that the hydrodynamics determined here provides a good basis for applications of fluid dynamics to granular flows.

The Chapman-Enskog method does not make explicit use of the form of the collision operator until a late stage in the analysis. Therefore, the results can be extended in several directions without new conceptual difficulties. One generalization of importance for practical applications is to include hard objects of different shapes, a degree of roughness, and a possible velocity dependence of α. In the opposite direction, it is of interest to consider simpler collision operators that allow access to solutions for more complex nonequilibrium states far from equilibrium. One such example has been explored in Sec. V. It is shown there that a simple kinetic model has the capacity to reproduce most of the n^* and α dependence of the transport coefficients, giving credibility to its application to more complex states. One example of the latter is a recent study of the rheology of a granular medium under shear flow [19]. Good agreement with Monte Carlo simulations of the RET are obtained for a wide range of n^*, α, and the shear rate.

As noted in the Introduction there are concerns that the qualitative changes introduced by inelastic collisions may invalidate the necessary conditions for a hydrodynamic or kinetic theory description. If the RET is a good description, the validity of hydrodynamics can be tested by Monte Carlo simulation of the evolution of initial states with small spatial gradients to check the n^* and α dependence of the transport properties predicted here. Further, a test of the validity of the RET is possible using molecular dynamics simulations. The calculations here, appropriate at high densities, make comparisons of transport properties in this manner as well. It is hoped that such simulations will be performed in the near future.

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APPENDIX A: CHAPMAN-ENSKOG EXPANSION

To initiate the Chapman-Enskog expansion it is useful to make a formal transformation of the RET to expose the effects of cooling. First, define the operator $J_E'[f]$ by

$$J_E'[f]=J_E[f]-\frac{1}{2}\zeta\boldsymbol{\nabla}_v\cdot(\mathbf{V}f), \tag{A1}$$

and decompose it according to

$$J_E'[f]\to\mathcal{P}J_E'[f]+\mathcal{Q}J_E'[f], \tag{A2}$$

where $\mathcal{P}$ is the projection operator onto an orthonormal set constructed from $\{1,v^2,\mathbf{v}\}$:

$$\mathcal{P}g(\mathbf{v})=n^{-1}\sum_\alpha\psi_\alpha(\mathbf{v})f^{(0)}(\mathbf{v})\int d\mathbf{v}'\,\psi_\alpha(\mathbf{v}')g(\mathbf{v}'), \tag{A3}$$

$$\{\psi_\alpha(\mathbf{V})\}=\left\{1,c^{-1/2}\left(\frac{m\beta}{2}V^2-\frac{3}{2}\right),(m\beta)^{1/2}\mathbf{V}\right\}. \tag{A4}$$

The normalization coefficient

$$c=n^{-1}\int d\mathbf{v}f^{(0)}(\mathbf{v})\left(\frac{m\beta}{2}V^2-\frac{3}{2}\right)^2=\frac{3}{2}+\frac{15}{8}c^*, \tag{A5}$$

with c^* given by Eq. (52). Also $\mathcal{Q}=(1-\mathcal{P})$ is the orthogonal projection. The contributions from $\mathcal{P}J_E'[f]$ can be calculated directly in terms of the collisional parts of the fluxes and the RET equation becomes

$$\begin{aligned}(\partial_t+\mathbf{v}\cdot\boldsymbol{\nabla})f-\frac{1}{2}\zeta\boldsymbol{\nabla}_v\cdot(\mathbf{V}f)&\\ =\mathcal{Q}J_E'[f]-\frac{\beta}{n}f^{(0)}\Bigg[\mathbf{V}\boldsymbol{\nabla}:\mathsf{P}^c+\frac{3}{2c}\left(\frac{m\beta}{3}V^2-1\right)&\\ \times(\boldsymbol{\nabla}\cdot\mathbf{q}^c+\mathsf{P}^c:\boldsymbol{\nabla}\mathbf{U})\Bigg].&\end{aligned} \tag{A6}$$

To zeroth order in the gradients this equation becomes

$$\mathcal{Q}\left[J^{(0)}[f^{(0)},f^{(0)}]-\frac{1}{2}\zeta^{(0)}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})\right]=0, \tag{A7}$$

which is the same as Eq. (19) of Sec. III.

To first order in the gradients (A6) becomes

$$\partial_t^{(0)} f^{(1)} = \mathcal{P}\frac{1}{2}\zeta^{(0)}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(1)}) + \mathcal{Q}J^{(1)}[f] - \mathcal{Q}\frac{1}{2}\zeta^{(1)}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)}) - \left[(\partial_t^{(1)}+\mathbf{v}\cdot\boldsymbol{\nabla})f^{(0)} - \frac{1}{2}\zeta^{(1)}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})\right] - \frac{\beta}{n}f^{(0)}\left[\mathbf{V}\cdot\boldsymbol{\nabla}p^c + \frac{3p^c}{2c}\left(\frac{m\beta}{3}V^2-1\right)\boldsymbol{\nabla}\cdot\mathbf{U}\right]. \tag{A8}$$

The first term on the right side vanishes, i.e.,

$$\mathcal{P}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(1)}) = 0, \tag{A9}$$

which follows from the condition $\mathcal{P}f^{(1)}=0$. The fourth term is calculated using the form (18)

$$\left[(\partial_t^{(1)}+\mathbf{v}\cdot\boldsymbol{\nabla})f^{(0)} - \frac{1}{2}\zeta^{(1)}\boldsymbol{\nabla}_v(\mathbf{V}f^{(0)})\right] = f^{(0)}(\partial_t^{(1)}+\mathbf{v}\cdot\boldsymbol{\nabla})\ln n - (\boldsymbol{\nabla}_{v_i}f^{(0)})(\partial_t^{(1)}+\mathbf{v}\cdot\boldsymbol{\nabla})U_i - \frac{1}{2}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})(\partial_t^{(1)}+\mathbf{v}\cdot\boldsymbol{\nabla})\ln T - \frac{1}{2}\zeta^{(1)}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)}). \tag{A10}$$

The macroscopic balance equations to first order in the gradients are

$$(\partial_t^{(1)}+\mathbf{U}\cdot\boldsymbol{\nabla})n + n\boldsymbol{\nabla}\cdot\mathbf{U} = 0, \tag{A11}$$

$$(\partial_t^{(1)}+\mathbf{U}\cdot\boldsymbol{\nabla})U_i + (mn)^{-1}\partial_i p = 0, \tag{A12}$$

$$(\partial_t^{(1)}+\mathbf{U}\cdot\boldsymbol{\nabla})T + \frac{2p}{3nk_B}\boldsymbol{\nabla}\cdot\mathbf{U} = -T\zeta^{(1)}. \tag{A13}$$

Use of these in Eq. (A10) gives finally

$$(\partial_t^{(1)}+\mathbf{v}\cdot\boldsymbol{\nabla})f^{(0)} - \frac{1}{2}\zeta^{(1)}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)}) = f^{(0)}\mathbf{V}\cdot\boldsymbol{\nabla}\ln n + (mn)^{-1}(\boldsymbol{\nabla}_v f^{(0)})\cdot\boldsymbol{\nabla}p - \frac{1}{2}[\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})]\mathbf{V}\cdot\boldsymbol{\nabla}\ln T - (\boldsymbol{\nabla}_{v_i}f^{(0)})(\mathbf{V}\cdot\boldsymbol{\nabla})U_i - \left[f^{(0)} - \frac{p}{3p^k}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})\right]\boldsymbol{\nabla}\cdot\mathbf{U}. \tag{A14}$$

Finally, the collision operator must be expanded to first order in the gradients. To do so the following results are needed:

$$\chi(\mathbf{r}_1,\mathbf{r}_1\pm\boldsymbol{\sigma}|n)\to\chi\left(1\pm\frac{1}{2}n\partial_n\ln\chi\,\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\ln n\right), \tag{A15}$$

$$f^{(0)}(\mathbf{r}_1\pm\boldsymbol{\sigma},\mathbf{v}_2,t) \to f^{(0)}(\mathbf{r}_1,\mathbf{v}_2,t)\left\{1\pm\left[\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\ln n - (\boldsymbol{\nabla}_{v_i}\ln f^{(0)})(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})U_i - \frac{1}{2}(3+\mathbf{V}\cdot\boldsymbol{\nabla}_v\ln f^{(0)})\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\ln T\right]\right\}, \tag{A16}$$

where χ is obtained from the functional $\chi(\mathbf{r}_1,\mathbf{r}_1\pm\boldsymbol{\sigma},|n)$ by evaluating all density fields at $n(\mathbf{r}_1,t)$. The collision operator to first order then becomes

$$\mathcal{Q}J_E[f]\to\mathcal{Q}J^{(0)}[f^{(0)},f^{(0)}] - \mathcal{L}f^{(1)} - \left(1+\frac{1}{2}n\partial_n\ln\chi\right)\boldsymbol{\mathcal{K}}[f^{(0)}]\cdot\boldsymbol{\nabla}\ln n + \mathcal{K}_j[\boldsymbol{\nabla}_{v_i}f^{(0)}]\partial_j U_i + \frac{1}{2}\boldsymbol{\mathcal{K}}[\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})]\cdot\boldsymbol{\nabla}\ln T, \tag{A17}$$

$$J^{(0)}[f^{(0)},f^{(0)}] = \chi\sigma^2\int d\mathbf{v}_2\int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}) \times\{\alpha^{-2}f^{(0)}(\mathbf{v}_1')f^{(0)}(\mathbf{v}_2') - f^{(0)}(\mathbf{v}_1)f^{(0)}(\mathbf{v}_2)\}, \tag{A18}$$

$$\mathcal{L}(X) \equiv -\mathcal{Q}(J^{(0)}[f^{(0)},X] + J^{(0)}[X,f^{(0)}]), \tag{A19}$$

$$\boldsymbol{\mathcal{K}}[X] = \mathcal{Q}\chi\sigma^2\int d\mathbf{v}_2\int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}) \times(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})\hat{\boldsymbol{\sigma}}\{\alpha^{-2}f^{(0)}(\mathbf{v}_1')X(\mathbf{v}_2') + f^{(0)}(\mathbf{v}_1)X(\mathbf{v}_2)\}. \tag{A20}$$

Substitution of Eqs. (A14) and (A17) into Eq. (A8) gives

$$\begin{aligned}(\partial_t^{(0)}+\mathcal{L})f^{(1)} + \mathcal{Q}\frac{1}{2}\zeta^{(1)}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)}) &= -\left\{\frac{p}{p^k}\left(1+n\partial_n\ln\frac{p}{p^k}\right)[f^{(0)}\mathbf{V}+(\beta m)^{-1}(\boldsymbol{\nabla}_v f^{(0)})] + \left(1+\frac{1}{2}n\partial_n\ln\chi\right)\boldsymbol{\mathcal{K}}[f^{(0)}]\right\}\cdot\boldsymbol{\nabla}\ln n \\ &+\left\{-[f^{(0)}\mathbf{V}+(\beta m)^{-1}(\boldsymbol{\nabla}_v f^{(0)})]\frac{p}{p^k} + f^{(0)}\mathbf{V} + \frac{1}{2}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})\mathbf{V} + \frac{1}{2}\boldsymbol{\mathcal{K}}[\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})]\right\}\cdot\boldsymbol{\nabla}\ln T \\ &+\left\{(V_j\boldsymbol{\nabla}_{v_i}f^{(0)}) - \frac{1}{3}\delta_{ij}\mathbf{V}\cdot\boldsymbol{\nabla}_v f^{(0)} + \mathcal{K}_j[\boldsymbol{\nabla}_{v_i}f^{(0)}] - \delta_{ij}\frac{p-p^k}{3p^k}\mathcal{Q}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})\right\}\partial_j U_i.\end{aligned} \tag{A21}$$

In the last line use has been made of the identity

$$\mathcal{P}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})=-f^{(0)}\frac{12}{4+5c^*}\left(\frac{m\beta}{3}V^2-1\right). \tag{A22}$$

To expose the functional dependence of $\zeta^{(1)}$ on $f^{(1)}$ Eq. (10) is evaluated directly to first order in the gradients

$$\begin{aligned}
\zeta=&(1-\alpha^2)\frac{\beta m\sigma^2\chi}{12n}\int d\mathbf{V}_1\int d\mathbf{V}_2\, f^{(0)}(V_1)f^{(0)}(V_2)\int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})^3\\
&+(1-\alpha^2)\frac{\beta m\sigma^2\chi}{12n}\int d\mathbf{V}_1\int d\mathbf{V}_2\, f^{(0)}(V_1)f^{(0)}(V_2)\int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})^3(\boldsymbol{\nabla}_{v_{2i}}\ln f^{(0)})(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})U_i\\
&+(1-\alpha^2)\frac{\beta m\sigma^2\chi}{12n}\int d\mathbf{V}_1\int d\mathbf{V}_2[f^{(0)}(V_1)f^{(1)}(V_2)+f^{(1)}(V_1)f^{(0)}(V_2)]\int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})^3\\
=&(1-\alpha^2)\frac{\beta m\pi\sigma^2\chi}{24n}\int d\mathbf{V}_1\int d\mathbf{V}_2\, f^{(0)}(V_1)f^{(0)}(V_2)g^3-(1-\alpha^2)\frac{\beta m\pi\sigma^2\chi}{18n}\int d\mathbf{V}_1\int d\mathbf{V}_2 f^{(0)}(V_1)f^{(0)}(V_2)g^2\boldsymbol{\nabla}\cdot\mathbf{U}\\
&+(1-\alpha^2)\frac{\beta m\pi\sigma^2\chi}{12n}\int d\mathbf{V}_1\int d\mathbf{V}_2\, f^{(0)}(V_1)f^{(1)}(V_2)g^3\\
=&\zeta^{(0)}-\frac{1}{3}(1-\alpha^2)\pi n^*\chi\boldsymbol{\nabla}\cdot\mathbf{U}+\zeta^{(1,1)}(f^{(1)}),
\end{aligned} \tag{A23}$$

$$\zeta^{(0)}\equiv(1-\alpha^2)\frac{\beta m\pi\sigma^2\chi}{24n}\int d\mathbf{V}_1\int d\mathbf{V}_2\, f^{(0)}(V_1)f^{(0)}(V_2)g^3, \tag{A24}$$

$$\zeta^{(1,1)}(f^{(1)})\equiv(1-\alpha^2)\frac{\beta m\pi\sigma^2\chi}{12n}\int d\mathbf{V}_1\int d\mathbf{V}_2\, f^{(0)}(V_1)f^{(1)}(V_2)g^3. \tag{A25}$$

With these results (A21) becomes

$$\begin{aligned}
&(\partial_t^{(0)}+\mathcal{L})f^{(1)}+\zeta^{(1,1)}(f^{(1)})\mathcal{Q}\frac{1}{2}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})\\
&\quad=-\left\{\frac{p}{p^k}\left(1+n\partial_n\ln\frac{p}{p^k}\right)[f^{(0)}\mathbf{V}+(\beta m)^{-1}(\boldsymbol{\nabla}_v f^{(0)})]+\left(1+\frac{1}{2}n\partial_n\ln\chi\right)\boldsymbol{\mathcal{K}}[f^{(0)}]\right\}\cdot\boldsymbol{\nabla}\ln n\\
&\quad\quad+\left\{-[f^{(0)}\mathbf{V}+(\beta m)^{-1}(\boldsymbol{\nabla}_v f^{(0)})]\frac{p}{p^k}+f^{(0)}\mathbf{V}+\frac{1}{2}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})\mathbf{V}+\frac{1}{2}\boldsymbol{\mathcal{K}}[\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})]\right\}\cdot\boldsymbol{\nabla}\ln T\\
&\quad\quad+\{(V_j\boldsymbol{\nabla}_{v_i}f^{(0)})+\mathcal{K}_j[\boldsymbol{\nabla}_{v_i}f^{(0)}]\}\frac{1}{2}\left(\partial_j U_i+\partial_i U_j-\frac{2}{3}\delta_{ij}\boldsymbol{\nabla}\cdot\mathbf{U}\right)\\
&\quad\quad+\left\{\frac{1}{3}\mathcal{K}_i[\boldsymbol{\nabla}_{v_i}f^{(0)}]+\frac{p-p^k}{3p^k}(2-3\alpha)\mathcal{Q}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})\right\}\boldsymbol{\nabla}\cdot\mathbf{U}.
\end{aligned} \tag{A26}$$

This is the result (22) used in the text.

APPENDIX B: SOLUBILITY CONDITIONS

The solubility conditions (Fredholm alternative [14]) for Eqs. (32)–(35) can be identified by using the property (36) and the definitions (32)–(35) to write these equations in the equivalent form

$$\mathcal{Q}\left(-\zeta^{(0)}T\partial_T+\mathcal{L}-\frac{\zeta^{(0)}}{2}\right)\boldsymbol{\mathcal{A}}=\mathbf{A}, \tag{B1}$$

$$\mathcal{Q}(-\zeta^{(0)}T\partial_T+\mathcal{L})\boldsymbol{\mathcal{B}}=[\mathbf{B}+(1+n\partial_n\ln\chi)\zeta^{(0)}\boldsymbol{\mathcal{A}}], \tag{B2}$$

$$\mathcal{Q}(-\zeta^{(0)}T\partial_T+\mathcal{L})\mathcal{C}_{ij}=C_{ij}, \tag{B3}$$

$$\mathcal{Q}(-\zeta^{(0)}T\partial_T+\mathcal{L})\mathcal{D}+\frac{1}{2}\zeta^{(1,1)}(\mathcal{D})\mathcal{Q}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})=D. \tag{B4}$$

Use has been made of the property

$$\mathcal{Q}T\partial_T\begin{pmatrix}\boldsymbol{\mathcal{A}}\\ \boldsymbol{\mathcal{B}}\\ \mathcal{C}_{ij}\\ \mathcal{D}\end{pmatrix}=-\mathcal{Q}\frac{1}{2}\boldsymbol{\nabla}_v\cdot\mathbf{V}\begin{pmatrix}\boldsymbol{\mathcal{A}}\\ \boldsymbol{\mathcal{B}}\\ \mathcal{C}_{ij}\\ \mathcal{D}\end{pmatrix}=T\partial_T\begin{pmatrix}\boldsymbol{\mathcal{A}}\\ \boldsymbol{\mathcal{B}}\\ \mathcal{C}_{ij}\\ \mathcal{D}\end{pmatrix} \tag{B5}$$

as follows from Eq. (A9). Since the left side of these equations lies in the orthogonal subspace it is necessary that the right sides must as well, or equivalently

$$\mathcal{P}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ C_{ij}\\ D\end{pmatrix}=0. \tag{B6}$$

These are the solubility conditions. It is straightforward to verify that they are satisfied by direct integration using the definitions (24)–(27) and Eq. (A3).

APPENDIX C: KINETIC CONTRIBUTIONS

Equations (48)–(50) for the kinetic contributions to the transport coefficients follow directly from the integral equations. Consider the viscosity. Multiply Eq. (34) by D_{ij} and integrate over the velocity to get

$$(-\zeta^{(0)}T\partial_T+\nu_\eta)\,\eta^k=-\frac{1}{10}\int d\mathbf{V}D_{ij}(\mathbf{V})C_{ij}, \tag{C1}$$

where ν_η is defined by Eq. (51). From dimensional analysis $\eta^k\propto T^{1/2}$ so

$$\eta^k=-\frac{1}{10[\nu_\eta-(1/2)\zeta^{(0)}]}\int d\mathbf{V}D_{ij}(\mathbf{V})C_{ij}(\mathbf{V}). \tag{C2}$$

The integral on the right side can be performed using the definition (26)

$$\begin{aligned}\frac{1}{10}\int d\mathbf{V}D_{ij}(\mathbf{V})C_{ij}(\mathbf{V})&=\frac{1}{10}\int d\mathbf{V}m\left(V_iV_j-\frac{1}{3}V^2\delta_{ij}\right)V_i(\boldsymbol{\nabla}_{v_j}f^{(0)})\\&\quad+\frac{1}{10}\int d\mathbf{V}D_{ij}(\mathbf{V})\mathcal{K}_i[\boldsymbol{\nabla}_{v_j}f^{(0)}]\\&=-p^k+\frac{1}{10}\int d\mathbf{V}D_{ij}(\mathbf{V})\mathcal{K}_i[\boldsymbol{\nabla}_{v_j}f^{(0)}],\end{aligned} \tag{C3}$$

and Eq. (C2) becomes

$$\eta^k=\frac{p^k}{\nu_\eta-\frac{1}{2}\zeta^{(0)}}\left[1-\frac{1}{10p^k}\int d\mathbf{V}D_{ij}(\mathbf{V})\mathcal{K}_i[\boldsymbol{\nabla}_{v_j}f^{(0)}]\right]. \tag{C4}$$

The remaining integral is performed using the definition of $\boldsymbol{\mathcal{K}}$ in Eq. (28)

$$\begin{aligned}\int d\mathbf{V}D_{ij}(\mathbf{V})\mathcal{K}_i[\boldsymbol{\nabla}_{v_j}f^{(0)}]&=\chi\sigma^2\int d\mathbf{V}_1D_{ij}(\mathbf{V}_1)\mathcal{Q}\int d\mathbf{V}_2\int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})\\&\quad\times(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})\sigma_i\{\alpha^{-2}f^{(0)}(\mathbf{V}_1')\boldsymbol{\nabla}_{v_{2j}'}f^{(0)}(\mathbf{V}_2')\\&\quad+f^{(0)}(\mathbf{V}_1)\boldsymbol{\nabla}_{v_{2j}}f^{(0)}(\mathbf{V}_2)\}.\end{aligned} \tag{C5}$$

The projection operator $\mathcal{Q}$ can be replaced by the identity operator because $D_{ij}(\mathbf{V}_1)$ is traceless. The scattering law for $\mathbf{V}_1'$ and $\mathbf{V}_2'$ is given in Sec. II. However, a simpler form is obtained by changing variables to integrate over $\mathbf{V}_1'$ and $\mathbf{V}_2'$ instead of $\mathbf{V}_1$ and $\mathbf{V}_2$ in the first term of Eq. (C5). The Jacobian of the transformation is α and $\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}=-\alpha\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}'$. Also, $\mathbf{V}_1(\mathbf{V}_1',\mathbf{V}_2')\equiv\mathbf{V}_1''=\mathbf{V}_1-\frac{1}{2}(1+\alpha)\hat{\boldsymbol{\sigma}}(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})$. The integral then becomes

$$\int d\mathbf{V} D_{ij}\mathcal{K}_i[\nabla_{v_j} f^{(0)}] = -\chi\sigma^2 \int d\mathbf{V}_1 \int d\mathbf{V}_2\, f^{(0)}(\mathbf{V}_1)[\nabla_{v_{2j}} f^{(0)}(\mathbf{V}_2)] \int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})\sigma_i[D_{ij}(\mathbf{V}_1'') - D_{ij}(\mathbf{V}_1)]$$

$$= \frac{1}{6} m(1+\alpha)\chi\sigma^3 \int d\mathbf{V}_1 \int d\mathbf{V}_2\, f^{(0)}(\mathbf{V}_1)[\nabla_{v_{2j}} f^{(0)}(\mathbf{V}_2)]$$

$$\times \int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})^2\{\hat{\sigma}_j[(\hat{\boldsymbol{\sigma}}\cdot\mathbf{V}_1) - (1+\alpha)(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})] + 3V_{1j}\}$$

$$= \frac{1}{6} m(1+\alpha)\chi\sigma^3 \int d\mathbf{V}_1 \int d\mathbf{V}_2\, f^{(0)}(\mathbf{V}_1) f^{(0)}(\mathbf{V}_2) \int d\hat{\boldsymbol{\sigma}}(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})\left\{4(\hat{\boldsymbol{\sigma}}\cdot\mathbf{V}_1) - \frac{3}{2}(1+\alpha)(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})\right\}$$

$$= \frac{1}{3}(1+\alpha) m\pi\chi\sigma^3 \int d\mathbf{V}_1 \int d\mathbf{V}_2\, f^{(0)}(\mathbf{V}_1) f^{(0)}(\mathbf{V}_2)\left[\frac{8}{3}(\mathbf{V}_1\cdot\mathbf{g}) - (1+\alpha)g^2\right]$$

$$= -\frac{2}{3} p^k \pi n^* \chi(1+\alpha)(3\alpha - 1). \tag{C6}$$

Use of this in Eq. (C4) gives the final result

$$\eta^k = \frac{p^k}{\nu_\eta - \frac{1}{2}\zeta^{(0)}}\left[1 + \frac{1}{15}\pi n^*\chi(1+\alpha)(3\alpha-1)\right]. \tag{C7}$$

The kinetic part of the thermal conductivity is obtained in a similar way. Multiplication of Eq. (32) by $\mathbf{S}(\mathbf{V})$ and integration over the velocity leads to

$$\kappa^k = -\frac{1}{3T}(\nu_\kappa - 2\zeta^{(0)})^{-1} \int d\mathbf{V}\mathbf{S}(\mathbf{V})\cdot\mathbf{A}(\mathbf{V})$$

$$= -\frac{1}{3T}(\nu_\kappa - 2\zeta^{(0)})^{-1} \int d\mathbf{V}\mathbf{S}(\mathbf{V})\cdot\left[\left(\frac{5}{2} f^{(0)} + \frac{1}{2}\mathbf{V}\cdot\nabla_v f^{(0)}\right)\mathbf{V} - \frac{p}{p^k}(\mathbf{V} f^{(0)} + (\beta m)^{-1}\nabla_v f^{(0)})\right]$$

$$- \frac{1}{6T}(\nu_\kappa - 2\zeta^{(0)})^{-1} \int d\mathbf{V}\mathbf{S}(\mathbf{V})\cdot\mathcal{K}[\nabla_v\cdot(\mathbf{V} f^{(0)})]$$

$$= \frac{5k_B p^k}{2m}(\nu_\kappa - 2\zeta^{(0)})^{-1}\left[1 + \frac{1}{2}\left(1 + \frac{p}{p^k}\right)c^* - \frac{m\beta}{15p^k}\int d\mathbf{V}\mathbf{S}(\mathbf{V})\cdot\mathcal{K}[\nabla_v\cdot(\mathbf{V} f^{(0)})]\right]. \tag{C8}$$

The last term on the right side is, more explicitly,

$$\int d\mathbf{V}\mathbf{S}(\mathbf{V})\cdot\mathcal{K}[\nabla_v\cdot(\mathbf{V} f^{(0)})]$$

$$= -\chi\sigma^2 \int d\mathbf{V}_1 \int d\mathbf{V}_2\, f^{(0)}(\mathbf{V}_1) \times\{\nabla_{v_2}\cdot[\mathbf{V}_2 f^{(0)}(\mathbf{V}_2)]\} \int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})\sigma_i \mathcal{Q}^*[S_i(\mathbf{V}_1'') - S_i(\mathbf{V}_1)]. \tag{C9}$$

Here $\mathcal{Q}^*$ is the adjoint projection operator with the effect

$$\mathcal{Q}^* S_i(\mathbf{V}_1) = S_i(\mathbf{V}_1) - \frac{1}{3\beta m} V_i \int d\mathbf{V} f^{(0)}(V)\mathbf{V}\cdot\mathbf{S}(\mathbf{V}) = V_i\left[\frac{1}{2} m V^2 - \frac{5}{2\beta}\left(1 + \frac{1}{2}c^*\right)\right]. \tag{C10}$$

With this result (C9) becomes

$$\int d\mathbf{V}\mathbf{S}(\mathbf{V})\cdot\mathcal{K}[\nabla_v\cdot(\mathbf{V} f^{(0)})]$$

$$= \frac{1}{2} m(1+\alpha)\chi\sigma^3 \int d\mathbf{V}_1 \int d\mathbf{V}_2\, f^{(0)}(\mathbf{V}_1) f^{(0)}(\mathbf{V}_2) \times\mathbf{V}_2\cdot\nabla_{v_2} \int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})^2\left[(\hat{\boldsymbol{\sigma}}\cdot\mathbf{V}_1)^2 - \frac{3}{4}(1+\alpha)(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{V}_1) + \frac{1}{8}(1+\alpha)^2(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})^2 + \frac{1}{2}V_1^2 - \frac{5}{2\beta m}\left(1 + \frac{1}{2}c^*\right)\right]$$

$$= -\frac{3}{2(\beta m)} p^k (1+\alpha)^2 \pi\chi n^*\left[-1 + 2\alpha + \left(\frac{1}{2}(1+\alpha) - \frac{5}{3(1+\alpha)}\right)c^*\right]. \tag{C11}$$

Use of this in Eq. (C8) gives the desired result

$$\kappa^k=\frac{5k_Bp^k}{2m}(\nu_\kappa-2\zeta^{(0)})^{-1}\left\{1+\frac{1}{2}\left(1+\frac{p}{p^k}\right)c^*+\frac{1}{10}(1+\alpha)^2\right.$$
$$\left.\times\pi\chi n^*\left[-1+2\alpha+\left(\frac{1}{2}(1+\alpha)-\frac{5}{3(1+\alpha)}\right)c^*\right]\right\}. \tag{C12}$$

To evaluate μ^k, multiply Eq. (33) by $\mathbf{S}(\mathbf{V})$ and integrate over the velocity to get

$$\mu^k=-(2\nu_\mu-3\zeta^{(0)})^{-1}\frac{1}{3n}\int d\mathbf{V}\,\mathbf{S}(\mathbf{V})\cdot[(1+n\partial_n\ln\chi)$$
$$\times\zeta^{(0)}\boldsymbol{\mathcal{A}}+\mathbf{B}]$$
$$=2(2\nu_\mu-3\zeta^{(0)})^{-1}\left[(1+n\partial_n\ln\chi)\zeta^{(0)}\frac{T}{n}\kappa^k\right.$$
$$\left.-\frac{1}{3n}\int d\mathbf{V}\,\mathbf{S}\cdot\mathbf{B}\right]$$
$$=2(2\nu_\mu-3\zeta^{(0)})^{-1}\left[(1+n\partial_n\ln\chi)\zeta^{(0)}\frac{T}{n}\kappa^k\right.$$
$$+\frac{5}{4}\frac{p}{p^k}\frac{1}{m\beta^2}\left(1+n\partial_n\ln\frac{p}{p^k}\right)c^*+\left(1+\frac{1}{2}n\partial_n\ln\chi\right)$$
$$\left.\times\frac{1}{3n}\int d\mathbf{V}\mathbf{S}(\mathbf{V})\cdot\boldsymbol{\mathcal{K}}[f^{(0)}]\right]. \tag{C13}$$

The last integral is

$$\int d\mathbf{V}\mathbf{S}(\mathbf{V})\cdot\boldsymbol{\mathcal{K}}[f^{(0)}]=-\chi\sigma^2\int d\mathbf{V}_1\int d\mathbf{V}_2\,f^{(0)}(\mathbf{V}_1)f^{(0)}(\mathbf{V}_2)\int d\hat{\boldsymbol{\sigma}}\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})\sigma_i\mathcal{Q}^*[S_i(\mathbf{V}_1'')-S_i(\mathbf{V}_1)]$$
$$=\frac{1}{2}m(1+\alpha)\chi\sigma^3\int d\mathbf{V}_1\int d\mathbf{V}_2\,f^{(0)}(\mathbf{V}_1)f^{(0)}(\mathbf{V}_2)\int d\hat{\boldsymbol{\sigma}}\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})^2\left[(\hat{\boldsymbol{\sigma}}\cdot\mathbf{V}_1)^2\right.$$
$$\left.-\frac{3}{4}(1+\alpha)(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{V}_1)+\frac{1}{8}(1+\alpha)^2(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})^2+\frac{1}{2}V_1^2-\frac{5}{2\beta m}\left(1+\frac{1}{2}c^*\right)\right]$$
$$=-\frac{3}{2\beta m}p^k(1+\alpha)\pi n^*\chi\left[\alpha(1-\alpha)+\frac{1}{4}\left(\alpha(1-\alpha)+\frac{4}{3}\right)c^*\right]. \tag{C14}$$

Substitution of Eq. (C14) into Eq. (C13) gives

$$\mu^k=2\frac{T}{n}(2\nu_\mu-3\zeta^{(0)})^{-1}\left\{(1+n\partial_n\ln\chi)\zeta^{(0)}\kappa^k+\frac{5}{4}\frac{pk_B}{m}\left(1+n\partial_n\ln\frac{p}{p^k}\right)c^*-\left(1+\frac{1}{2}n\partial_n\ln\chi\right)\right.$$
$$\left.\times\frac{k_Bp^k}{2m}(1+\alpha)\pi n^*\chi\left[\alpha(1-\alpha)+\frac{1}{4}\left(\alpha(1-\alpha)+\frac{4}{3}\right)c^*\right]\right\}. \tag{C15}$$

APPENDIX D: COLLISIONAL TRANSFER CONTRIBUTIONS

The collisional transfer contributions to the pressure tensor and heat flux are determined from Eqs. (8) and (9). Consider first the pressure tensor which becomes to first order in the gradients

$$P_{ij}^c=\frac{1}{4}(1+\alpha)m\sigma^3\chi\int d\mathbf{V}_1\int d\mathbf{V}_2\int d\hat{\boldsymbol{\sigma}}\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})^2\hat{\sigma}_i\hat{\sigma}_j\left[f^{(0)}(\mathbf{V}_1)f^{(1)}(\mathbf{V}_2)+f^{(1)}(\mathbf{V}_1)f^{(0)}(\mathbf{V}_2)\right.$$
$$\left.-\frac{1}{2}f^{(0)}(\mathbf{V}_2)\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nabla}f^{(0)}(\mathbf{V}_1)+\frac{1}{2}f^{(0)}(\mathbf{V}_1)\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nabla}f^{(0)}(\mathbf{V}_2)\right]$$
$$=\frac{1}{2}(1+\alpha)m\sigma^3\chi\int d\mathbf{V}_1\int d\mathbf{V}_2\int d\hat{\boldsymbol{\sigma}}\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})^2\hat{\sigma}_i\hat{\sigma}_j\left[f^{(1)}(\mathbf{V}_1)f^{(0)}(\mathbf{V}_2)+\frac{1}{2}f^{(0)}(\mathbf{V}_1)\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nabla}f^{(0)}(\mathbf{V}_2)\right]$$
$$=\frac{1}{15}(1+\alpha)m\sigma^3\pi\chi\int d\mathbf{V}_1\int d\mathbf{V}_2\,f^{(1)}(\mathbf{V}_1)f^{(0)}(\mathbf{V}_2)(2g_ig_j+g^2\delta_{ij})$$
$$-\partial_kU_\ell\frac{1+\alpha}{4}m\sigma^4\chi\int d\mathbf{V}_1\int d\mathbf{V}_2\,f^{(0)}(\mathbf{V}_1)\boldsymbol{\nabla}_{v_{2\ell}}f^{(0)}(\mathbf{V}_2)\int d\hat{\boldsymbol{\sigma}}\Theta(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})^2\hat{\sigma}_i\hat{\sigma}_j\hat{\sigma}_k$$

$$
\begin{aligned}
&=\frac{2}{15}(1+\alpha)n^*\pi\chi P_{ij}^k+\partial_k U_\ell\frac{1+\alpha}{48}m\sigma^4\pi\chi\int d\mathbf{V}_1\int d\mathbf{V}_2\, f^{(0)}(\mathbf{V}_1)f^{(0)}(\mathbf{V}_2)\\
&\quad\times\boldsymbol{\nabla}_{v_{2\ell}}g^{-1}[g_ig_jg_k+g^2(g_j\delta_{ik}+g_i\delta_{jk}+g_k\delta_{ij})]\\
&=\frac{2}{15}(1+\alpha)mn^*\pi\chi P_{ij}^k+\partial_k U_\ell\frac{1+\alpha}{48}m\sigma^4\pi\chi\int d\mathbf{V}_1\int d\mathbf{V}_2\, f^{(0)}(\mathbf{V}_1)f^{(0)}(\mathbf{V}_2)\\
&\quad\times\{g^{-3}g_\ell g_ig_jg_k-g^{-1}(g_jg_k\delta_{i\ell}+g_ig_k\delta_{j\ell}+g_jg_i\delta_{k\ell}+g_\ell g_j\delta_{ik}+g_\ell g_i\delta_{jk}+g_\ell g_k\delta_{ij})-g(\delta_{j\ell}\delta_{ik}+\delta_{i\ell}\delta_{jk}+\delta_{k\ell}\delta_{ij})\}\\
&=\frac{2}{15}(1+\alpha)n^*\pi\chi P_{ij}^k-\frac{1+\alpha}{18}\frac{m\sigma\pi\chi n^*}{n}\int d\mathbf{V}_1\int d\mathbf{V}_2\, f^{(0)}(\mathbf{V}_1)f^{(0)}(\mathbf{V}_2)g\left[\frac{3}{5}\left(\partial_jU_i+\partial_iU_j-\frac{2}{3}\delta_{ij}\boldsymbol{\nabla}\cdot\mathbf{U}\right)+\delta_{ij}\boldsymbol{\nabla}\cdot\mathbf{U}\right].
\end{aligned}
\tag{D1}
$$

From this equation one can easily identify the collisional transfer contributions to the shear viscosity η and the bulk viscosity γ given by Eqs. (38), (39), and (45).

The collisional transfer contribution to the heat flux to first order in the gradients can be obtained in a similar way. The result is

$$
\begin{aligned}
q_i^c&=\frac{1}{2}(1+\alpha)m\sigma^3\chi\int d\mathbf{V}_1\int d\mathbf{V}_2\int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})^2(\hat{\boldsymbol{\sigma}}\cdot\mathbf{G})\hat{\sigma}_i\left[f^{(1)}(\mathbf{V}_1)f^{(0)}(\mathbf{V}_2)+\frac{1}{2}f^{(0)}(\mathbf{V}_1)\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nabla}f^{(0)}(\mathbf{V}_2)\right]\\
&=\frac{1}{5}(1+\alpha)n^*\pi\chi q_i^k-\partial_iT\frac{1+\alpha}{48}\frac{m\sigma\pi\chi n^*}{nT}\int d\mathbf{V}_1\int d\mathbf{V}_2\, f^{(0)}(\mathbf{V}_1)f^{(0)}(\mathbf{V}_2)\left[g^{-1}(\mathbf{g}\cdot\mathbf{G})^2+gG^2+\frac{3}{2}g(\mathbf{g}\cdot\mathbf{G})+\frac{1}{4}g^3\right].
\end{aligned}
\tag{D2}
$$

From Eq. (D2) one may identify the collisional contributions to the thermal conductivity κ and the coefficient μ given in Eqs. (40), (41), and (46).

APPENDIX E: FIRST SONINE APPROXIMATION

In this appendix an outline of the integrations appearing in the evaluation of the transport coefficients is given using the first Sonine approximations for the zeroth-order distribution (53) and for the first-order distribution (56). The frequencies ν_η^* and $\nu_\kappa^*=\nu_\mu^*$ are given by Eq. (61) in the first approximation. It is straightforward to show that the projection operator $\mathcal{Q}$ in the definition of $\mathcal{L}$ can be replaced by the identity in both integrations, and these integrals become the same as those appearing in the Boltzmann limit (except for a factor χ) [1]. The details will not be repeated here and only the results are quoted in Table I.

The collisional contributions are given in terms of the dimensionless integrals I_γ and I_κ defined by Eqs. (45) and (46), respectively. In both cases, the integrations over the relative and center of mass variables $\mathbf{g}$ and $\mathbf{G}$ are performed and nonlinear terms in c^* are neglected. The integral I_γ is given by

$$
\begin{aligned}
I_\gamma&=\pi^{-3}v_0^{-7}\int d\mathbf{g}\int d\mathbf{G}e^{-g^2/2v_0^2}e^{-2G^2/v_0^2}g\\
&\quad\times\left\{1+\frac{c^*}{4}\left[\frac{15}{2}-5v_0^{-2}\left(\frac{g^2}{2}+2G^2\right)\right.\right.\\
&\quad\left.\left.+2v_0^{-4}(\mathbf{g}\cdot\mathbf{G})^2+2v_0^{-4}\left(\frac{g^2}{4}+G^2\right)^2\right]\right\}\\
&=\frac{4}{(2\pi)^{1/2}}\left(1-\frac{1}{32}c^*\right).
\end{aligned}
\tag{E1}
$$

In a similar way, the integral I_κ is given by

$$
\begin{aligned}
I_\kappa&=\pi^{-3}v_0^{-9}\int d\mathbf{g}\int d\mathbf{G}e^{-g^2/2v_0^2}e^{-2G^2/v_0^2}\\
&\quad\times\left\{1+\frac{c^*}{4}\left[\frac{15}{2}-5v_0^{-2}\left(\frac{g^2}{2}+2G^2\right)\right.\right.\\
&\quad\left.\left.+2v_0^{-4}(\mathbf{g}\cdot\mathbf{G})^2+2v_0^{-4}\left(\frac{g^2}{4}+G^2\right)^2\right]\right\}\\
&\quad\times\left[g^{-1}(\mathbf{g}\cdot\mathbf{G})^2+gG^2+\frac{3}{2}g(\mathbf{g}\cdot\mathbf{G})+\frac{1}{4}g^3\right]\\
&=\frac{8}{(2\pi)^{1/2}}\left(1+\frac{7}{32}c^*\right).
\end{aligned}
\tag{E2}
$$

APPENDIX F: CONTRIBUTIONS TO THE COOLING RATE

Up to the first order in gradients, the cooling rate is

$$
\zeta=\zeta^{(0)}-\frac{1}{3}(1-\alpha^2)\pi n^*\chi\boldsymbol{\nabla}\cdot\mathbf{U}+\zeta^{(1,1)},
\tag{F1}
$$

where $\zeta^{(0)}$ and $\zeta^{(1,1)}$ are defined by Eqs. (16) and (30), respectively. The contribution $\zeta^{(0)}$ can be evaluated easily by using the first Sonine approximation to $f^{(0)}$ [1]. Its expression appears in Table I. The two first contributions to the cooling rate were calculated by Goldshtein and Shapiro [7],

although an error was made in the expression for c^*. The corrected value for the latter was obtained in Ref. [13] and is given by Eq. (55). With this change the results of Ref. [7] agree with the first two terms on the right side of Eq. (F1).

In the first Sonine approximation, $\zeta^{(1,1)}$ is given by

$$\zeta^{(1,1)}(\mathcal{D})=(1-\alpha^2)\frac{\beta m\pi\sigma^2\chi}{12n}c_{\mathcal{D}}\int d\mathbf{V}_1\int d\mathbf{V}_2 \times f^{(0)}(\mathbf{V_1})f_M(\mathbf{V}_2)E(V_2)g^3 =\frac{1}{2}(1-\alpha^2)\left(1+\frac{3}{64}c^*\right)\left(\frac{\pi}{m\beta}\right)^{1/2}n\sigma^2\chi c_{\mathcal{D}}\ . \tag{F2}$$

In order to determine $c_{\mathcal{D}}$, multiply the integral equation (35) by $E(V)$ and integrate over $\mathbf{V}$ to get

$$(-\zeta^{(0)}T\partial_T+\nu_\gamma)c_{\mathcal{D}} =\frac{2}{15n}\int d\mathbf{V}E(V)D(\mathbf{V}) -\frac{1}{15n}\zeta^{(1,1)}\int d\mathbf{V}E(V)\mathcal{Q}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)}), \tag{F3}$$

with the definition

$$\nu_\gamma=\frac{\int d\mathbf{V}E(V)\mathcal{L}[f_M(\mathbf{V})E(V)]}{\int d\mathbf{V}f_M(\mathbf{V})E(V)E(V)}. \tag{F4}$$

Dimensional analysis requires that $\zeta^{(0)}$ and ν_γ are proportional to $T^{1/2}$ while the integrals on the right-hand side are independent of T. Therefore $c_{\mathcal{D}}\propto T^{-1/2}$ and Eq. (F3) gives

$$c_{\mathcal{D}}=\frac{2}{15n}\frac{1}{\frac{1}{2}\zeta^{(0)}+\nu_\gamma}\left[\int d\mathbf{V}E(V)D(\mathbf{V}) -\frac{1}{2}\zeta^{(1,1)}\int d\mathbf{V}E(V)\mathcal{Q}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})\right]. \tag{F5}$$

To convert Eq. (F5) in an explicit expression for $c_{\mathcal{D}}$ the two integrals appearing in the right-hand side of Eq. (F5) and the quantity ν_γ must be evaluated. The calculation is straightforward, leading to the results

$$\int d\mathbf{V}E(V)\mathcal{Q}\boldsymbol{\nabla}_v\cdot(\mathbf{V}f^{(0)})=\frac{15}{2}nc^*(\alpha), \tag{F6}$$

$$\int d\mathbf{V}E(V)\mathcal{K}_i[\boldsymbol{\nabla}_{v_i}f^{(0)}] =\frac{3}{8}\left(5\alpha^2+4\alpha-1+\frac{c^*}{12}\frac{159\alpha+3\alpha^2-19\alpha-15\alpha^3}{1-\alpha}\right) \times(1-\alpha^2)\pi n^*\chi n\equiv\lambda(\alpha)\pi n^*\chi n, \tag{F7}$$

$$\nu_\gamma^*=\frac{\nu_\gamma}{\nu_0}=\frac{1+\alpha}{48}\chi\left[128-96\alpha+15\alpha^2-15\alpha^3 +\frac{c^*}{64}(15\alpha^3-15\alpha^2+498\alpha-434)\right], \tag{F8}$$

and consequently

$$c_{\mathcal{D}}=\frac{1}{\nu_0}\frac{(2/45)\lambda\pi n^*\chi+[(p-p^k)/3p^k](2-3\alpha)c^*}{\frac{1}{2}\zeta^{(0)*}+\nu_\gamma^*+(5c^*/64)\left(1+\frac{3}{64}c^*\right)\chi(1-\alpha^2)}. \tag{F9}$$

Finally, $\zeta^{(1,1)}$ is obtained from Eqs. (F2) and (F9). We are not aware of any previous calculation of $\zeta^{(1,1)}$ for comparison. It is noted that $\zeta^{(1,1)}$ vanishes in the limits of elastic spheres ($\alpha=1$, arbitrary density) and of dilute inelastic spheres ($n^*=0$, arbitrary inelasticity).

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